Brownian beads

Bálint Virág*

June 7, 2003

Abstract

We show that the past and future of half-plane Brownian motion at certain cutpoints are independent of each other after a conformal transformation. Like in Itô's excursion theory, the pieces between cutpoints form a Poisson process with respect to a local time. The size of the path as a function of this local time is a stable subordinator whose index is given by the exponent of the probability that a stretch of the path has no cutpoint. The index is computed and equals 1/2.

1 Introduction

The mathematical theory of conformally invariant planar stochastic models has seen great progress in the recent years. The goal of this paper is to consider Brownian motion, the first example of a conformally invariant process, and further explore its conformal structure.

Let B denote Brownian motion started at zero and conditioned to stay in the upper half plane; we call this distribution BE, or half-plane excursion. This is a transient process, and its path has cut-points (Burdzy, 1989), that is points which, if removed, make the image of B disconnected. We call the segments of the paths between consecutive cutpoints **Brownian beads**.

For a given cutpoint, the complement of the past path has one infinite connected component. This can be mapped back to the half plane via a conformal homeomorphism. It is convenient to normalize this map so that it has derivative 1 at ∞ and takes the cutpoint to 0; we call this map the **conformal shift**. Our first goal is to show that Brownian beads are

*Department of Mathematics, MIT, Cambridge, MA 02139, USA. balint@math.mit.edu. Research partially supported by NSF grant #DMS-0206781.

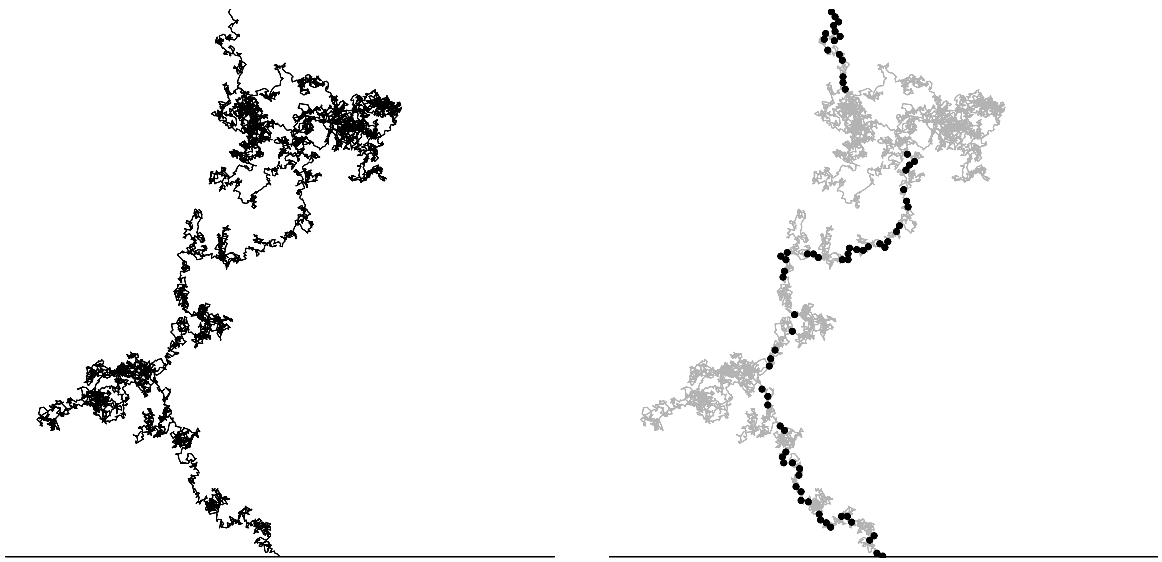

Figure 1: Brownian excursion in $\mathbb{H}$ and its cutpoints

independent of each other after a conformal transformation. More precisely, for a cutpoint g let f denote the corresponding conformal shift and let β the bead starting at g (if any).

Theorem 1. *There exists a local time λ supported on cutpoints so that $(f_\lambda(\beta_\lambda), \lambda \geq 0)$ is a Poisson point process.*

Here f_λ applied to a path maps the image of the path according to the conformal map, and changes time-parameterization according to local Brownian scaling (Section 4). The intensity measure of the Poisson point process is a σ-finite measure on paths conditioned to start without cutpoints; we call this measure the bead process. In Section 6 we prove a Markov property for the bead process and show that it has finite lifetime.

Theorem 1 may be thought of as a two-dimensional analogy of Itô's theorem about Brownian excursions. Recall that excursions of 1-dimensional Brownian motion are the segments of the path between two consecutive visits to 0. For a zero g' let β' denote the excursion starting at g', and let f' denote the map that shifts the future of the path by $-g'$.

Theorem 2 (Itô). *There exists a local time λ' on zeros so that $(f'_{\lambda'}(\beta'_{\lambda'}), \lambda' \geq 0)$ is a Poisson point process.*

The local time λ' at zero has the nice property that the $(g'(\lambda'), \lambda' \geq 0)$ process, or in other words the total duration of excursions up to the zero g' is a stable subordinator of index 1/2. It turns out that the analogous statement holds in our setting. Let $a(t)$ denote the half-plane capacity of the path up to time t; half-plane capacity is a way to measure the size of subsets of $\mathbb{H}$ (see Section 4). Then

Theorem 3. *The process* $(a \circ g(\lambda), \lambda \geq 0)$ *is a stable subordinator of index* α*.*

Another property of beads which is shared by Itô excursions is that the process $(\beta_\lambda,\ \lambda \geq 0)$ determines $(B(t),\ t \geq 0)$ (see Remark 21).

In Section 7 we show that the index α can be expressed as an exponent.

Theorem 4. *We have* $\mathbf{P}(B \text{ has no cuttime in } (1,t)) = t^{-\alpha+o(1)}$*. as* $t \to \infty$*.*

Finally, we compute α.

Theorem 5. *The Brownian bead index* α *equals* $1/2$*.*

Although the 1/2 here is equal to the index for Itô's excursions, this seems to be a coincidence. Neither is it a consequence of Brownian scaling. Unlike the proof of most exponents, our proof of Theorem 5 does not rely on SLE processes (nor exponents that have been determined using these processes). It uses beads, not the exponent representation; I am grateful to W. Werner with whom I discussed ideas of this kind. A related exponent governing the Hausdorff dimension of cuttimes for SLE_6 was computed by Beffara (2003), and it seems possible to derive the value of α directly from his results. Werner (2003) also computed an exponent related to α using $\mathrm{SLE}(\kappa, \rho)$ processes (in a paper with beautiful ideas and computations but few rigorous proofs). The advantage of the proof here is that it is more elementary and conceptual; it relies only on the special intersection exponent $\xi(2,1)$, whose value was determined by Lawler (1995).

In the proof, we will use the following fact about Brownian excursion. Let A be a compact subset of $\overline{\mathbb{H}} \setminus \{0\}$ so that the sets $\mathbb{R} \cup A$ and $\mathbb{H} \setminus A$ are connected. Let $f : \mathbb{H} \setminus A \to \mathbb{H}$ be a conformal homeomorphism fixing ∞ with $f'(\infty) = 1$.

Proposition 6. $\mathbf{P}(B \text{ avoids } A) = f'(0)$*.*

Further interesting properties of Brownian motion have recently been found using the fact that its outer boundary has the same distribution as (the outer boundary of) some SLE processes, see Lawler, Schramm, and Werner (2003) and the references therein. Some of methods used in this paper appear independently in the recent literature; in particular, see Dubedat (2003) and Lawler and Werner (2003).

Many of the arguments of this paper carry over to ordinary Brownian motion, more precisely, Brownian excursion from a boundary point to an interior point of a domain. The analysis there is a bit more difficult since there is no scale invariance; we do not follow this avenue here.

In Section 2 we discuss some simple properties of Brownian excursions in planar domains that we will need later. We also show Proposition 6 there. In Section 3 we introduce the cuttime filtration and prove a Markov property with respect to this filtration. Section 4 reviews the facts we need from conformal geometry, and introduces some basic semigroups of paths. In Section 5 we prove the Poisson point process decomposition of Theorem 1, and we define the bead process, a Brownian excursion in the half plane conditioned to start without cutpoints. In the next section we show that this special process does not need infinite time to start off, a fact that is not clear from the definition. In the last two sections, 7 and 8, we prove Theorems 4 and 5; we also give some open questions and conjectures.

2 Properties of Brownian excursion

Let D be a **regular domain**, i.e. a simply connected open subset of the plane whose boundary is locally connected. Let ∂D denote the Caratheodory boundary, i.e. the set of prime ends of D.

For $\{a\}, Z \subset D \cup \partial D$, and $a \neq \infty$, let $\mathsf{BE}(a, Z, D)$ denote Brownian motion started at a and conditioned to hit Z no later than ∂D. If this event has positive probability, then this definition is precise; otherwise it can be made precise by considering h-processes or by taking a limit. A special case is $\mathsf{BE}(0, \infty, \mathbb{H})$, the half-plane excursion, which we will often abbreviate BE. The coordinates of BE are Brownian motion and an independent 3-dimensional Bessel process.

The two most important properties of $B \sim \mathsf{BE}(a, Z, D)$ are restriction and conformal invariance. Restriction says that if $D' \subset D$, then on the event that B stays in D', it has distribution $\mathsf{BE}(a, Z', D')$, where $Z' = Z \cap (D' \cup \partial D')$. Conformal invariance says that if $f : (a, Z, D) \to (a', Z', D')$ is a conformal homeomorphism between two regular domains, then after a time change (22) the image of B under f has distribution $\mathsf{BE}(a', Z', D')$. The restriction property is straightforward; conformal invariance has been first proved by Lévy (1940); the later development of stochastic calculus makes it a simple exercise.

It is important for the previous paragraph that by Theorem 9.8 in the book of Pommerenke (1975), conformal homeomorphisms from the half plane $\mathbb{H}$ or unit disk to regular domains extend continuously to the boundary.

Remark 7. One way to get such domains is the following. Let E denote the image of a curve $\gamma : [0,1] \to \mathbb{C}$, and let D be a connected component of $\mathbb{C} \setminus E$. Since local connectivity is preserved under continuous maps, E is locally connected. The proof of Theorem 9.8 in Pommerenke (1975), part (ii) $\to$ (iii) carries through without changes to show that in fact the boundary of D is also locally connected.

The strong Markov property says that the future of B after a stopping time τ given $\mathcal{F}_\tau$ has distribution $\mathsf{BE}(B(\tau), Z, D)$.

The following theorem is due to Cranston and McConnell (1983) (see Chung (1984) for a simpler proof).

Theorem 8. *There exists a universal constant c so that the lifetime τ of a path distributed $\mathsf{BE}(a, Z, D)$ has $\mathbf{E}\tau < c$ area(D).*

We say that a sequence $a_n \in D$ **converges to** $a \in \partial D$ **along a path** if there is a continuous curve $\gamma : (0,1) \to D$ so that $a_n = \gamma(t_n)$ for some sequence t_n, and γ is contained in the equivalence class that defines the prime end a.

In order to define a distance, we may extend paths defined on a finite interval $[0,t]$ to be constant on $(-\infty, 0]$ and $[t, \infty)$. Let

$$\begin{aligned} \mathrm{dist}_0(\pi_1, \pi_2) &= \sup_{t \in \mathbb{R}} \|\pi_1(t) - \pi_2(t)\|, \\ \mathrm{dist}(\pi_1, \pi_2) &= \inf_{\varepsilon \in \mathbb{R}} \left(|\varepsilon| + \mathrm{dist}_0(\theta_\varepsilon \pi_1, \pi_2) \right). \end{aligned}$$

where θ_ε denotes time shift. Let "$\Rightarrow$" denote convergence in distribution (in the case of paths with respect to the metric "dist").

Proposition 9. *If $a_n \to a \in \partial D$ along a path, then $\mathsf{BE}(a_n, Z, D) \Rightarrow \mathsf{BE}(a, Z, D)$.*

Proof. Consider the conformal homeomorphism $f : D \to \mathbb{H}$ which takes the prime end a (as determined by the path of convergence of a_n) to 0. Since the domain D is regular, we have $f(a_n) \to 0$. Let

$$N_\varepsilon = \{f^{-1}(z) \ : \ z \in \mathbb{H}, |z| < \varepsilon\}.$$

By regularity of D, the map f^{-1} extends continuously to the boundary, and so it is uniformly continuous in a neighborhood of 0. It follows that $\mathrm{diam}(N_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now consider the processes in the half-plane. Trying to couple at hitting times of circles of radius 2^{-k} about 0, one can easily check that the two processes can be coupled before hitting the circle of radius ε with probability at least

$$1 - p_{n,\varepsilon} = 1 - c(|f(a_n)|/\varepsilon)^\gamma,$$

where $c, \gamma > 0$ are absolute constants. Let C_n denote the event that the processes in the general domain D can be coupled before their first escape times τ_ε and $\tau_{n,\varepsilon}$ from N_ε. By conformal invariance, $\mathbf{P}(C_n) \geq 1 - p_{n,\varepsilon}$. Let B, B_n denote the coupled processes on the event C_n.

Let A_n be the event that $|\tau_\varepsilon - \tau_{n,\varepsilon}| < \delta$. By Theorem 8 and Markov's inequality, $\mathbf{P}(A_n^c) < c_2 \operatorname{area}(N_\varepsilon)/\delta$, for some universal constant c_2. Now if A_n and C_n hold, and $\varepsilon < \delta$, then $\operatorname{dist}(B, B_n) < 3\delta$. Therefore

$$\mathbf{P}(\operatorname{dist}(B, B_n) > 3\delta) \leq c_2 \operatorname{area}(N_\varepsilon)/\delta + p_{n,\varepsilon} + \mathbf{1}(\delta \leq \varepsilon)$$

and the right hand side can be made arbitrarily small by first picking ε then letting $n \to \infty$. It follows that B_n converges to B in probability and so in distribution. □

Corollary 10. *Let $Z_n \to 1$ be sets of reals, and let $a_n \in \mathbb{H}$, $a_n \to 0$. Then*

$$\mathsf{BE}(a_n, Z_n, \mathbb{H}) \Rightarrow \mathsf{BE}(0, 1, \mathbb{H}).$$

Proof. It suffices to prove this for the case when each $Z_n = \{z_n\}$ is a single point, since the general case is a mixture of such processes. Consider the rescaled versions $\mathsf{BE}(a_n/z_n, 1, \mathbb{H})$ of $\mathsf{BE}(a_n, z_n, \mathbb{H})$. These converge to $\mathsf{BE}(0, 1, \mathbb{H})$ by the lemma, and since the scaling factor converges to 1, so do the original processes. □

Let $\mathsf{BE}(a, z, D, K)$ denote the distribution of $\mathsf{BE}(a, z, D)$ conditioned to hit K. Say a sequence of sets $A_n \subset D$ **converges to** a **along a path** if every sequence of points $a_n \in A_n$ does.

Lemma 11. *Let D be a regular domain, and consider regular subdomains $D_n = D \setminus K_n$, where the $K_n \subset D$ are relatively closed. Let $L_n \subset D_n$ closed, and let $a_n \in D_n$. Suppose that*

$$M_n := K_n \cup L_n \cup \{a_n\} \to \{a\} \subset \partial D \quad \textit{along a path.}$$

Then

$$\mathsf{BE}(a_n, z, D_n, L_n) \Rightarrow \mathsf{BE}(a, z, D).$$

Proof. Let W_n have distribution $\mathsf{BE}(a_n, z, D_n, L_n)$. The slightly difficult part is to construct relatively closed sets $S_n \to \{a\}$ separating z from M_n in D so that

$$\mathbf{P}(W_n \text{ hits } S_n \text{ before } L_n) \to 0, \tag{1}$$

$$\sup_{s\in S_n} \mathbf{P}(\mathsf{BE}(s,z,D) \text{ hits } K_n) \to 0. \tag{2}$$

To complete the proof from here, let τ_n be the hitting time of S_n for W_n. Since $S_n \to \{a\}$, we have $\tau_n \to 0$. If we condition W_n to have done its duty of hitting L_n by time τ_n, then $\{W_n(\tau_n + t),\ t \geq 0\}$ has the same distribution as $\mathsf{BE}(W(\tau_n), z, D_n)$. This, in turn has the same distribution as $\mathsf{BE}(W(\tau_n), z, D)$ conditioned not to hit K_n. In both cases, we are conditioning on events whose probabilities converge to 1. Also, $\mathsf{BE}(W(\tau_n), z, D)$ is a mixture of processes starting from points of S_n, so by Proposition 9 it converges to $\mathsf{BE}(a, z, D)$. It follows that the distribution of W_n converges to $\mathsf{BE}(a, z, D)$, as required.

Now we proceed to find the sets S_n. By conformal invariance of the probabilities involved, it suffices to do this in the upper half plane with $a = 0,\ z = \infty$. Here we use that our domain has a curve boundary, so if in $\mathbb{H}$ we have $S_n \to 0$, then for its image under the conformal homeomorphism $(0, \infty, \mathbb{H}) \to (a, z, D)$ we have $S'_n \to a$.

Let $c_n \to \infty$ slow enough that we still have $c_n M_n \to 0$; if we construct S_n that are bounded for this rescaled problem, then scaling back will make $S_n \to 0$. So it suffices to construct bounded S_n. We consider the uniformizing map g_n which takes D_n to the half plane $\mathbb{H}$. Assume that g_n has hydrodynamic normalization, and extend it to the boundary of D_n. The half-plane version of the Caratheodory Kernel Theorem (see Theorem 1.8 in Pommerenke (1975)) implies that $g_n(z) \to z$ uniformly where it is defined. So if we $L'_n = g_n(L_n)$, $a'_n = g_n(a_n)$ then $L'_n \cup \{a'_n\} \to \{0\}$.

Let S be the unit semicircle in $\mathbb{H}$, and use the shorthand $B_n = \mathsf{BE}(a'_n, \infty, \mathbb{H})$. We will show that

$$\mathbf{P}(B_n \text{ hits } S \text{ before } L'_n \mid B_n \text{ hits } L'_n) \to 0, \tag{3}$$

$$\sup_{s\in S} \mathbf{P}(\mathsf{BE}(s, \infty, \mathbb{H}) \text{ hits } L'_n) \to 0. \tag{4}$$

Once we have this, we take $S_n = g_n^{-1}(S)$ so by uniform convergence, $S_n \to S$. Thus $\{S_n\}$ is bounded, and $(1, 2)$ follow from conformal invariance.

The second statement (4) is obvious, and the first follows since we can easily find semicircles $R_n \to \{0\}$ so that

$$\sup_{w\in R_n} \mathbf{P}(\mathsf{BE}(w, \infty, \mathbb{H}) \text{ hits } L'_n) < \mathbf{P}(\mathsf{BE}(a'_n, \infty, \mathbb{H}) \text{ hits } L'_n). \tag{5}$$

The left hand side of (3) can be written as

$$\mathbf{P}(B_n \text{ hits } S \text{ before } L'_n \text{ and hits } L'_n) \ /\ \mathbf{P}(B_n \text{ hits } L'_n). \tag{6}$$

Using the Markov property and the topology of the setup the numerator can be bounded above by

$$\sup_{w\in S}\mathbf{P}(\mathsf{BE}(w,\infty,\mathbb{H}) \text{ hits } R_n) \sup_{w\in R_n}\mathbf{P}(\mathsf{BE}(w,\infty,\mathbb{H}) \text{ hits } L_n') \tag{7}$$

and so by (5), expression (6) is bounded above by the first factor of (7), which converges to 0, proving (3). □

Lemma 12. *Let D be a regular domain, and let $K \subset D$ be a relatively compact connected set containing at least two points. Let $z \in \partial D$. Suppose that $K_n \subset D$ are relatively compact and converge to K. Suppose that $a_n \to a$. Then*

$$\mathsf{BE}(a,z,D,K_n) \Rightarrow \mathsf{BE}(a_n,z,D,K).$$

PROOF. Let g be a bounded continuous test function for paths, and let B_n, B be distributed as $\mathsf{BE}(a_n,z,D)$ and $\mathsf{BE}(a,z,D)$ respectively.

Note that for $\mathsf{BE}(a,z,D)$-almost all paths π (1) the function $f_n = \mathbf{1}(\pi \textit{ hits } K_n)$ converges to $f = \mathbf{1}(\pi \textit{ hits } K)$ and (2) f is continuous at π. Thus by Proposition 9 we have

$$\begin{aligned}\mathbf{E}(f_n(B_n)g(B_n)) &\to \mathbf{E}(f(B)g(B)),\\ \mathbf{E}f_n(B_n) &\to \mathbf{E}f(B).\end{aligned}$$

Note that the hypothesis of the lemma ensures that $\mathbf{E}f(B)$ is positive. Taking the ratio of the previous two limit statements we get the desired result. □

Hitting probabilities

Let D be a domain, and let a, z be points on $\partial D \setminus A$ in the neighborhood of which the boundary is a differentiable curve. Let A be a hull in $\overline{D}$ not containing a, z. We show the following equivalent version of Proposition 6.

Proposition 13. *Let f be a conformal homeomorphism that takes $D\setminus A$ to D and fixes a, z. Then*

$$\mathbf{P}(\mathsf{BE}(a,z,D) \textit{ avoids } A) = f'(a)f'(z).$$

PROOF. Because of conformal invariance, it is sufficient to prove this in the half-plane $\mathbb{H}$ with $a = 0$, $z = 1$.

The indicator function of the event above is continuous at almost every path with respect to the excursion measure. Let $a_n = i/n$, and $Z_n = (1, 1 + 1/n) \subset \mathbb{R}$. It suffices to show that as $n \to \infty$ we have

$$\mathbf{P}(\mathsf{BE}(a_n,Z_n,\mathbb{H}) \text{ avoids } A) \to f'(0)f'(1) \tag{8}$$

since the measures on the left converge to the excursion measure by Corollary 10.

$$\mathbf{P}(\mathsf{BM}(a) \text{ hits } Z_n \text{ no later than } \partial D) = h_{1/n}(i/n) = \nu n^{-2} + o(n^{-2}) \tag{9}$$

Here h_ε is the harmonic function determined by the boundary conditions 1 on $(1, 1+\varepsilon)$ and 0 elsewhere, and ν is some fixed constant. The functions h can be easily given explicitly, but the formulas are unimportant, all we need is the intuitive fact that as $\varepsilon, z \to 0$ we have

$$h_\varepsilon(z) = \nu\varepsilon\Im z + o(\varepsilon|z|)$$

giving the second equality in (9). Similarly, after a conformal transformation by f

$$\mathbf{P}(\mathsf{BM}(a_n) \text{ hits } Z_n \text{ no later than } A \cup \partial D) \tag{10}$$

becomes

$$\begin{aligned}\mathbf{P}(\mathsf{BM}(f(a_n)) \text{ hits } f(Z_n) \text{ no later than } f(A \cup \partial\mathbb{H})) \\ = h_{f(1+\varepsilon)-1}(f(i\varepsilon)) = f'(0)f'(1)\nu n^{-2} + o(n^{-2})\end{aligned}$$

but the ratio of (10) and (9) gives (8), proving the claim. □

3 The cuttime filtration

Consider the excursion $\{B_t\}$ with distribution $\mathsf{BE}(a, z, D)$ where D is a regular domain, $a, z \in \partial D$. Let G denote the set of **cuttimes**, that is, the set of times for which images of the future and past are disjoint; Burdzy (1989) showed that such times exist with probability 1. It is easy to check that the set G is measurable and it is a closed set a.s.

It is clear that one cannot decide what the cuttimes are up to time t by only looking at the past of the process $\{B_t\}$. $G \cap [0, t]$ is therefore not measurable with respect to the standard filtration $\mathcal{F}_t$ of $\{B_t\}$. We therefore introduce the **cuttime filtration** $\mathcal{G}_t$ generated by $\mathcal{F}_t$ and $G \cap [0, t]$.

This enlargement of filtration may look large, but in fact it is not in the following sense. Let $\hat{G}_t$ denote the set of cuttimes of the process $\{B_t\}$ restricted to the interval $[0, t]$. If τ is the earliest time for which the set of points $B[0, \tau]$ is revisited after time t, then we clearly have

$$G \cap [0, t] = \hat{G}_t \cap [0, \tau].$$

Since $\hat{G}_t$ has Hausdorff dimension less than one (Lawler, 1996), and the distribution of B_τ is absolutely continuous with respect to harmonic measure on $B[0, t]$, we have by Makarov's theorem that $\tau \notin \hat{G}_t$ $\mathcal{F}_t$-a.s. In particular, $G \cap [0,t]$ is determined by the connected component (g_t, g_t') of $[0,t] \setminus \hat{G}_t$ containing τ. The left endpoint of this interval, g_t, is the last cuttime up to time t. Then given $\mathcal{F}_t$, g_t determines $\mathcal{G}_t$; its distribution is concentrated on the countable set of points in $\hat{G}_t$ that are isolated from the right.

Let ρB_s denote the connected component of $D \setminus B[0,s]$ having z as an interior or boundary point. By Remark 7 if D is a regular domain, then so is ρB_s for every s. Let $\mathsf{BE}(a,z,D,K)$ denote the distribution of $\mathsf{BE}(a,z,D)$ conditioned to hit K.

Proposition 14 (Markov property for cuttime filtration). *Let τ be a $\mathcal{G}_t$-stopping time. Then the distribution of $\{B_{\tau+t}, t \geq 0\}$ given $\mathcal{G}_\tau$ is $\mathsf{BE}(B_t, z, \rho B_{g_t}, B[g_t, g_t'])$.*

Note that, while for fixed time τ, B_τ is an interior point of ρB_{g_τ} with probability one, there exists stopping times for which $g_\tau = \tau$ a.s. An example is the first cuttime after time 1. In these cases the "hitting" condition of the proposition holds trivially.

Proof of Proposition 14.

Recall that $\mathcal{G}_t$ is generated by $\mathcal{F}_t$ and g_t. Since the distribution of $\{B_{t+s}, s \geq 0\}$ given $\mathcal{F}_t$ is $\mathsf{BE}(B_t, z, D)$ by the Markov property, further conditioning on the value of g_t means that the future of the process $\mathsf{BE}(B_t, z, D)$ has to stay in ρB_{g_τ}, and hit $B[g_\tau, g_\tau']$. This completes the proof for fixed time t. Denote L_t the set $B[g_t, g_t']$, and denote $\mathbf{E}_t$ the distribution conditioned as above, that is $\mathsf{BE}(B_t, z, \rho B_t, L_t)$.

What we showed is equivalent to the following. If X is a bounded random variable on paths, and the time shift operator θ_t is defined as $(\theta_t B)(s) = B(s+t)$, then

$$\mathbf{E}(X \circ \theta_t \mid \mathcal{G}_t) = \mathbf{E}_t X.$$

Thus, if τ is a $\mathcal{G}_t$-stopping time concentrated on a discrete set of values $\mathcal{T}$, then

$$\begin{aligned} \mathbf{E}(X \circ \theta_\tau \mid \mathcal{G}_\tau) &= \sum_{t \in \mathcal{T}} \mathbf{E}(X \circ \theta_\tau \; \mathbf{1}(\tau = t) \mid \mathcal{G}_\tau) \;=\; \sum_{t \in \mathcal{T}} \mathbf{E}(X \circ \theta_t \; \mathbf{1}(\tau = t) \mid \mathcal{G}_t) \\ &= \sum_{t \in \mathcal{T}} \mathbf{1}(\tau = t) \mathbf{E}(X \circ \theta_t \mid \mathcal{G}_t) \;=\; \sum_{t \in \mathcal{T}} \mathbf{1}(\tau = t) \mathbf{E}_t X \;=\; \mathbf{E}_\tau X, \end{aligned}$$

which completes the proof for this case. For an arbitrary $\mathcal{G}_t$-stopping time τ we take a discrete approximation by letting τ_n be the first element of $2^{-n}\mathbb{Z}$ which is at least τ. Since the decision to stop at time τ_n is made by time τ, we have

$$\mathbf{E}(X \circ \theta_{\tau_n} \mid \mathcal{G}_{\tau_n}) = \mathbf{E}(X \circ \theta_{\tau_n} \mid \mathcal{G}_\tau).$$

Since θ_t is a.s. continuous as $t \downarrow 0$, and X is continuous, we have $X \circ \theta_{\tau_n} \to X \circ \theta_\tau$ almost surely, and by the bounded convergence theorem

$$\mathbf{E}(X \circ \theta_{\tau_n} \mid \mathcal{G}_\tau) \to \mathbf{E}(X \circ \theta_\tau \mid \mathcal{G}_\tau).$$

Therefore it suffices to prove that $\mathcal{F}_\infty$-a.s. $\mathbf{E}_{\tau_n} X \to \mathbf{E}_\tau X$, or

$$\mathsf{BE}(B_{\tau_n}, z, \rho B_{g_{\tau_n}}, L_{\tau_n}) \Rightarrow \mathsf{BE}(B_\tau, z, \rho B_{g_\tau}, L_\tau). \tag{11}$$

First we fix the path B and examine what happens to g_{τ_n} and g'_{τ_n} in the limit. The definition $g_\tau = \sup(G \cap [0, \tau])$ implies that g_τ is increasing in τ and continuous from the right. Assume that $\tau \neq g_\tau$. Then τ lies in a connected component of G whose left endpoint is g_τ, so for all large $n > N_1$ we must have $g_{\tau_n} = g_\tau$.

Now assume also that $g'_\tau \neq \tau$. Then $B[g_\tau, g'_\tau]$ is disjoint from $B(g'_\tau, \tau]$. Let t be the first time after τ that B_t revisits the set $B[g_\tau, g'_\tau]$. Then for $\tau_n < t$ and $n > N_1$ we have $g'_{\tau_n} = g'_\tau$. On the other hand, if $g'_\tau = \tau$, then $g'_{\tau_n} \leq \tau_n$ implies that $g'_{\tau_n} \downarrow g'_\tau$.

So whenever $\tau \neq g_\tau$, we have $\rho B_{g_{\tau_n}} = \rho B_{g_\tau}$ eventually, $L_{\tau_n} \to L_\tau$, and the latter contains at least two points. By Lemma 12 we get

$$\mathsf{BE}(B_{\tau_n}, z, \rho B_{g_\tau}, L_{\tau_n}) \Rightarrow \mathsf{BE}(B_\tau, z, \rho B_{g_\tau}, L_\tau),$$

which implies (11).

If $g_\tau = \tau$, then set $K_n := B[g_\tau, g_{\tau_n}]$. Note that $K_n \cup L_{\tau_n} \cup \{B_{\tau_n}\} \subset B[\tau, \tau_n]$ converges to $\{B_\tau\}$ along a path (i.e. a sub-path of B_t). Therefore Lemma 11 yields (11). □

4 Conformal maps, paths, semigroups

We begin this section with some notation and standard facts about complex geometry. Most can be found in Pommerenke (1975), Lawler (2001), and best of all the upcoming book Lawler (2003), which reviews complex geometry with an eye towards applications in stochastic processes.

If $A \subset \mathbb{H}$, let $h_y(A)$ be harmonic measure, or the hitting distribution of Brownian motion started at y and stopped at $A \cup \mathbb{R}$. Let

$$h_\infty(A) = \lim_{y \to \infty} y \cdot h_y(A)$$

Define

$$\operatorname{cap}_0(A) = h_\infty(A), \tag{12}$$
$$\operatorname{cap}_1(A) = \int_{\overline{A}} \Im(z) h_\infty(dz). \tag{13}$$

If D is a subset of $\mathbb{H}$ so that $\mathbb{H} \setminus D$ is bounded, then the capacity cap_0 in D can be defined analogously and is denoted $\text{cap}_0(D, A)$. Conformal invariance of harmonic measure implies that if f is a conformal homeomorphism $D \to \mathbb{H}$ satisfying $f'(\infty) = 1$, then $\text{cap}_0(D, A) = \text{cap}_0(f(A))$. The quantity cap_1 will be referred to as half-plane capacity.

If A is bounded, $A \cup \mathbb{R}$ is connected, and $f : \mathbb{H} \setminus A \to \mathbb{H}$ is a conformal homeomorphism with hydrodynamic normalization (i.e. $f(z) - z \to 0$ as $z \to \infty$), then we have

$$f(z) = z + \text{cap}_1(A)/z + O(1/z^2) \tag{14}$$

as $z \to \infty$. Such maps have the property that for all $z \in \mathbb{H}$

$$\Im f(z) \leq \Im(z). \tag{15}$$

Recall the following standard fact about complex geometry. There exist a constant c so that if $A \subset \overline{\mathbb{H}}$ is connected, then

$$c^{-1} \operatorname{diam}(A) \leq \text{cap}_0(A) \leq c \operatorname{diam}(A), \tag{16}$$

if $A \cup \mathbb{R}$ is connected and $x + iy \in A$, then

$$y^2/4 \leq \text{cap}_1(A). \tag{17}$$

Also, if A, A' are disjoint, then we have

$$\text{cap}_1(A \cup A') \leq \text{cap}_1(A) + \text{cap}_1(A') \tag{18}$$

This implies that if A_1, A_2, A are disjoint, with $A_1 \cup R$ and $\mathbb{H} \setminus A_1$ connected, then

$$\text{cap}_1(A_1 \cup A_2 \cup A) - \text{cap}_1(A_1 \cup A) \leq \text{cap}_1(A_1 \cup A_2) - \text{cap}_1(A_1) \tag{19}$$

We apply the conformal map $f : (\mathbb{H} \setminus A_1, 0, \infty) \to (\mathbb{H} \setminus A_1, 0, \infty)$; then by additivity of cap_1 the desired inequality transforms to $\text{cap}_1(f(A_2 \cup A)) - \text{cap}_1(f(A)) \leq \text{cap}_1(f(A_2))$, which follows from (18).

Let Φ denote the set of paths $\pi : [0, \tau] \to \overline{\mathbb{H}}$ which intersect the real line only at time 0 and $\pi(\tau)$ is in the boundary of $\rho\varphi$, the infinite connected component of the complement of the image $\pi[0, \tau]$.

To each $\pi \in \Phi$ we can associate the unique conformal homeomorphism $f_\pi : \mathbb{H} \to \rho_\pi$ with hydrodynamic normalization. which fixes ∞, has derivative 1 there, and extends to the boundary $\mathbb{R}$ continuously (see Remark 7) and maps 0 to $\pi(t)$. Such a map has expansion

$$f(z) = z + a_0 + az^{-1} + O(z^{-2}) \tag{20}$$

at ∞. The coefficient $a = \text{cap}_1(\pi)$ behaves additively under compositions of maps. It has the scaling property

$$a(r\pi) = r^2 a(\pi). \tag{21}$$

Using the conformal maps, it is possible to compose two paths in Φ. Let $\pi \circ \pi'$ equal π on $[0, \ell_\pi]$, and equal the f_π-transform of π' (see the definition below) time-shifted by ℓ_π for the rest of the time interval.

Thus Φ is a semigroup, and the map a is a semigroup homomorphism $\Phi \to [0, \infty)$.

Definition 15. Let π be a path in a regular domain D_1 (possibly starting and ending at ∂D_1), and let f be a conformal homeomorphism $D_1 \to D_2$. If the integral

$$s(t) = \int_0^t |f' \circ \pi(\tau)|^2 d\tau \tag{22}$$

is finite for each t, then let $t(s)$ be the inverse of $s(t)$; the function $f \circ \pi \circ t(s)$ is called the ***f*-transform** of $\pi(t)$.

5 Independent increments and a local time for beads

Consider the process $B \sim \mathsf{BE}$, and let G denote the set of cuttimes. If $B(t)$ is on the boundary of ρB_t, let f_t denote the normalized conformal homeomorphism mapping $(\rho B_t, B(t), \infty)$ to $(\mathbb{H}, 0, \infty)$. In this section, we explore the following consequence of Proposition 14 and the conformal invariance of Brownian excursion.

Corollary 16 (Independent increments at cuttimes). *Let τ be a $\mathcal{G}_t$-stopping time supported on the set of cuttimes G. Then the f_τ-map of the future has distribution* BE.

By a minor abuse of notation, let $a(t)$ denote the half-plane capacity of $B[0, t]$.

Lemma 17. *The set $a(G)$ is a closed regenerative set with a scale-invariant distribution.*

Recall that a random set S has scale-invariant distribution for every $c > 0$ the sets cS are identically distributed. It is called regenerative if for every $\{\sigma[S \cap [0, t]];\ t \geq 0\}$-stopping time ν which satisfies $\nu \in S$ a.s. the translated set $(S - \tau) \cap [0, \infty)$ has the same distribution as S.

Proof. Closedness follows from the continuity of half-plane capacity and since the set of cuttimes is closed. Scale-invariance follows from the scale-invariance of Brownian excursion and the scaling property (21) of the half-plane capacity a.

To check the regenerative property, assume that the stopping time ν is as above. Then $\tau = a^{-1}(\nu)$ is a stopping time for the cutpoint filtration $\mathcal{G}$, and $\tau = a^{-1}(\nu)$ is supported on cutpoints. Therefore by Corollary 16 the distribution of the f_τ-mapping of the future of B after τ has distribution BE.

In particular, since a behaves additively under conformal homeomorphisms, the distribution of $(a(G) - \nu) \cap [0, \infty)$ given the past of the excursion is the same as the distribution of $a(G)$. Thus $a(G)$ is a regenerative set, as required. □

By a result of Kingman (1973), the above implies that $a(G)$ is the image of a stable subordinator. More precisely, there exists a nondecreasing random function $(\lambda(a),\, a \geq 0)$ adapted to the filtration $(\sigma(a(G) \cap [0,a]),\, a \geq 0)$ so that λ increases exactly on the set $a(G)$ its right-continuous inverse $(a_\lambda,\, \lambda \geq 0)$ is a stable subordinator with index $\alpha \in (0,1)$. The normalization can be chosen so that the Lévy measure (the intensity of the Poisson point process of jumps) assigns mass $y^{-\alpha}$ to the interval (y, ∞) for all $y \geq 0$. We call λ the **bead (local) time**.

Let $B_{\to\lambda}$ denote the process $(B(t), 0 \leq t \leq t(\lambda))$ as an element of the semigroup Φ. Recall that a Lévy process X on a topological semigroup Φ is a right continuous process with left limits with the property that at any fixed time t, given the entire past $(X_s, 0 \leq s \leq t)$, the distribution of X is the same as the past composed with an independent copy of X. Since the inverse bead local times are $\mathcal{G}_t$-stopping times supported on G, Corollary 16 immediately gives

Proposition 18. *$(B_{\to\lambda},\, \lambda \geq 0)$ is a Φ-valued Lévy process.*

Let $t(\lambda)$ be the time corresponding to local time λ, i.e. the solution of $a(t(\lambda)) = a_\lambda$. Whenever the process a_λ has a jump, the segment $(B(t),\, t \in [t(\lambda-), t(\lambda+)])$ has no cutpoints, while $t(\lambda-)$, $t(\lambda+)$ are cuttimes. Let $\beta(\lambda)$ denote the f-mapping of this segment from $\rho B(t(\lambda-))$ to $\mathbb{H}$. Then $\beta(\lambda)$ is an element of the semigroup Φ. When $a(\lambda)$ has no jump, we set $\beta(\lambda)$ equal some null state.

Note the following deterministic fact. For an interval $I = [\lambda_1, \lambda_2]$ the set $\beta(I)$ equals the set $\beta'(I - \lambda_1)$ for the process which is the $f_{a(\lambda_1)}$-image of the original one. Since the inverse bead local times are stopping times, the independent increment property of Corollary 16 implies that for non-overlapping intervals I, the sets $\beta(I)$ are independent. We have shown

Proposition 19. *β is a Φ-valued Poisson point process.*

The **Brownian bead measure** (BB) is the σ-finite intensity measure of the Poisson point process β. In simple terms, for a set of paths A the measure $\mathsf{BB}(A)$ equals the expected number of elements of A among the beads $\beta[0,1]$.

6 Properties of beads

The goal of this section is to establish some simple properties of the **bead process**, i.e. the measure BB.

Scaling. It follows from the scaling properties of BE and half-plane capacity that for $c > 0$ we have $\mathsf{BB}(rA) = r^{-2a}\mathsf{BB}(A)$, where scaling a path by c means scaling in space by a factor c and time accordingly. This implies that the law BB can be decomposed as a product of measures on "shape space" and "size space". Let β be chosen according to BB conditioned to have size a at least 1, and rescale β to have size 1. The resulting probability measure NBB, determines the shape of BB, while the size is given independently by the measure $d(y^{-\alpha})$.

We now check that the Markov property for cuttimes and conformal invariance imply a Markov property of beads.

Note that the time of beads is only defined up to translation (since the integral in the f-transform may not be finite). We may pick a rule to set $t = t_1$ for some fixed number t_1 when (and if) the bead grows to size 1 (which here is an arbitrary positive number). Let t_0 be the starting time of the bead (possibly $-\infty$), and let A_t denote the image of the time interval between t_0 and the first t-local cuttime after t_0 under B. Let $T \geq t_1$ be a stopping time with respect to the canonical filtration generated by the past of the process.

Lemma 20 (Markov property of beads). *Under the measure* BB*, the process* $(B(T + s),\ s \geq 0)$ *has the same distribution as a process* B' *distributed as* $\mathsf{BE}(B(T), \infty, \mathbb{H}, A_T)$ *and stopped at a random time* τ*. Here* τ *is the first time that*

$$(B[t_0, T] \cup B'[0, \tau)) \cap B'(\tau, \infty] = 0.$$

Proof. It suffices to show this for the case $T = t_1$ (recall that the bead starts at a time $t_0 < t_1$). The general case follows from first applying the lemma at $T = t_1$ and then using the Markov property of Brownian excursion. More precisely, if B' is as given in the claim, and $T \geq t_1$ is a stopping time, then the distribution of the future of B after T is just the distribution of the future of B' after T, and $B' \sim \mathsf{BE}(B(T), \infty, \mathbb{H}, A_T)$ stopped at τ.

For the $T = t_1$ case note that BB conditioned to have size at least 1 is by definition the following. BE is run until the first time S that $a(S) - a(g'(S)) = 1$ where $g'(S), g(S)$

are the global cuttimes immediately before and after time S. Then BB is the $f_{g'(S)}$-map of $B(g'(S)+s,\, 0 \le s \le g(S)-g'(S))$.

Therefore the statement of the lemma follows from the Markov property of Proposition 14, conformal invariance and the fact that f-mappings preserve half-plane capacity. □

Note that an example of such a stopping time is the first time when BB hits the line with imaginary part y. This will be used in Lemma 23 to show that in fact beads have finite lifetime, and can be started at $t_0 = 0$. Once this has been done, it is straightforward to extend Lemma 20 to arbitrary stopping times $T > t_0$.

Remark 21 (Beads determine the excursion). Another property shared by beads and Itô excursions is that the process $(\beta_\lambda,\, \lambda \ge 0)$ determines the process $(B(t),\, t \ge 0)$. When $B(t)$ is on the boundary of $B[0,t]$ (these are the so-called pioneer points), the parameter a_0 in the conformal shift (20) defines the "horizontal" location of $B(t)$.

Given β, it is straightforward to determine a_0 as a function of the half-plane capacity a for each $a \ge 0$. First, $a(\lambda)$ is a stable subordinator and so its value is just the sum of its jumps. We can thus determine a as a function of λ. The arguments of this section and symmetry imply that $a_0(\lambda)$ is a symmetric stable process, so its value can also be determined as a sum of its jumps taken in the right order (i.e. summing all jumps of absolute size at least ε and then letting $\varepsilon \to 0$). Finally, the value of $a_0(a)$ between cutpoints can be easily determined by looking at the appropriate bead.

This gives a Löwner chain for the pioneer points of B (see, for example, Lawler (2001) for definitions), and hence determines its outer boundary. Now assume that the outer boundaries of B and B' agree, but they still differ in some way within a particular bead. Since f-mappings are one-to-one, then that bead has to be mapped by the same conformal shift to different points in the processes β, β'. This answers a question posed by an anonymous referee.

The proof of the following simple fact is left to the reader.

Lemma 22. *Let A be a nonempty subset of $\mathbb{R} \times (0,1] \subset \mathbb{H}$ so that $A \cup \mathbb{R}$ is connected. Let B have distribution $\mathsf{BE}(z,\infty,\mathbb{H})$ conditioned to hit A and stopped when this happens. Let T_1 be the time B spends in the strip with imaginary parts between 1 and 2. There exists absolute constants c, γ so that for all $t > 0$ we have*

$$\mathbf{P}(T > t) < ce^{-\gamma t}.$$

Lemma 23 (Finite lifetime). *Let T denote the lifetime of the process with distribution* BB*. Then a.e.* $T < \infty$*.*

Proof. Let B be a bead conditioned to hit $\Im z = y$. By the Markov property, the future of B after this time is just $\mathsf{BE}(z, \infty, \mathbb{H})$ conditioned to hit a certain subset of the past. Let $T(y, 2y)$ denote the time B spends with imaginary part in this interval. By Lemma 22 and scale invariance

$$\mathbf{P}(T(y, 2y) > ty^2) < ce^{-\gamma t}$$

and therefore, for the unconditioned measure

$$\mathsf{BB}(T(y, 2y) > ty^2) < y^{-\alpha} c' e^{-\gamma t}.$$

setting $y_n = 2^{-n}$, and $t_n = c_1 n$ the right hand side becomes $2^{\alpha n} 2^{-c_1 \gamma n}$ which is summable for an appropriate choice of c_1. By the first Borel-Cantelli lemma (which also holds for σ-finite measures) we get that BB a.e. for all large n

$$T(2^{-n}, 2^{1-n}) \leq cn2^{-2n}$$

summing this we get that for some "random" constant K and all $y \leq 1$

$$T(0, y) < Ky^2 |\log y| \tag{23}$$

But the Markov property and the existence of cutpoints for large times implies that after hitting $\Im(z) = 1$, B will have finite lifetime T' a.s. Therefore $T \leq T(0, 1) + T'$ is finite BB-a.e. as required. □

7 The exponent giving the bead index

The goal of this section is to identify the index α of the stable process driving Brownian beads as an exponent for a large deviation event. Let $A(t)$ denote the event that the half-plane excursion B has no cuttime between times 1 and t.

Theorem 4. *For large t, we have $\mathbf{P}A(t) = t^{-\alpha + o(1)}$.*

Werner (2003) recently computed essentially the same exponent using generalized SLE processes, with the result $\alpha = 1/2$. We will compute α directly by a simpler argument in the next section.

Let $(a(\lambda), \lambda \geq 0)$ be a stable subordinator with index α. We will use the following simple fact. It follows directly from Bertoin (1996) page 76 Proposition 2.

Fact 24. *Let*

$$X = \min\left(a([0,\infty)) \cap [1,\infty)\right)$$

then for some positive c as $x \to \infty$ we have $\mathbf{P}(X > x) \sim cx^{-\alpha}$.

Let $A'(a)$ denote the event that there is no cuttime so that the past has half-plane capacity between 1 and a. Fact 24 implies

$$\mathbf{P}A'(a) \sim ca^{-\alpha} \qquad \text{as } a \to \infty. \tag{24}$$

In order to conclude Theorem 4, we only need to show that half-plane capacity and time are not too far from each other; in fact it suffices to show the following.

Lemma 25. *We have $\mathbf{P}(t/s < \operatorname{cap}_1 B[0,t] < ts) \geq 1 - ce^{-\gamma s}$.*

Proof. For a set A in the plane, let M_x and M_y denote the sup of the absolute value of the projection of A to the x and y axes, respectively. If $A \subset \overline{\mathbb{H}}$ contains zero and is connected, then by (17) and considering the half-plane capacity of a rectangle we get

$$M_y^2/4 \leq \operatorname{cap}_1(A) \leq c\max(M_y^2, M_x).$$

Now let $A = B[0,1]$. Then it is easy to check the following simple property of the maxima of Brownian motion and the 3-dimensional Bessel processes.

$$\mathbf{P}(1/s < M_y^2 \leq \max(M_y^2, M_x) < s) \geq 1 - ce^{-\gamma s},$$

and the claim follows by scale-invariance. □

8 The value of the bead index α

Consider the process $B \sim \mathsf{BE}$; in this section we will index B either by time or bead local time λ; in the latter case we will stick to the notation λ. Let $\mathcal{L}$ denote Lebesgue measure, let μ denote the random measure on the half plane given by

$$\mu(A) = \mathcal{L}(\lambda : B(\lambda) \in A),$$

and let $\overline{\mu}(A) := \mathbf{E}\mu(A)$. By the scaling of half-plane capacity $a(\lambda)$ and the scale-invariance of Brownian motion we get that for $r \geq 0$

$$\overline{\mu}(rA) = r^{2\alpha}\overline{\mu}(A).$$

Lemma 26. *The $\overline{\mu}$-measure of $\mathbb{S}_1 = \mathbb{R} \times (0, 1]$ is finite.*

PROOF. Consider the random measure

$$\mu(\Lambda, A) = \mathcal{L}(\lambda \in \Lambda : B(\lambda) \in A)$$

and let $\overline{\mu}(\Lambda, A) := \mathbf{E}\mu(\Lambda, A)$. Let λ be the first bead time at least 1 so that $B(\lambda) \in \mathbb{S}_1$, and let f denote the corresponding conformal shift. Let $p = \mathbf{P}(\lambda < \infty)$; it is easy to check that $p < 1$. For $n \geq 0$ we have

$$\begin{aligned} \mathbf{E}(\mu([1, n+1], \mathbb{S}_1) \,|\, \mathcal{G}_\lambda) &\leq \mathbf{1}(\lambda < \infty)\overline{\mu}([0, n+1-\lambda], f(\mathbb{S}_1)) \\ &\leq \mathbf{1}(\lambda < \infty)\overline{\mu}([0, n], \mathbb{S}_1) \end{aligned}$$

because of the cuttime Markov property (Proposition 14) and the fact (15) that $f(\mathbb{S}_1) \subset \mathbb{S}_1$. Taking expected values gives

$$\overline{\mu}([1, n+1], \mathbb{S}_1) \leq p\overline{\mu}([0, n], \mathbb{S}_1) =: p\overline{\mu}_n.$$

This yields the recursion $\overline{\mu}_{n+1} \leq 1 + p\overline{\mu}_n$, which gives the bound $\overline{\mu}_\infty \leq 1/(1-p)$, as required. □

Lemma 27. *$\overline{\mu}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{H}$ with density bounded below and above on compacts.*

PROOF. Let $A \subset \mathbb{H} \setminus \{0\}$ be a closed set so that $A \cup \mathbb{R}$ and $\mathbb{H} \setminus A$ are connected, and let f be the conformal homeomorphism $(\mathbb{H} \setminus A, 0, \infty) \to (\mathbb{H}, 0, \infty)$. Call such f a subdomain map.

Fix a path B which avoids A, and consider its image $f(B)$. The cuttimes g of B can be parameterized by $a(g)$ i.e. cap_1 of the past, as well as $a'(g)$ that is cap_1 of the past of the image $f(B)$. Fact (19) implies that if $g_1 < g_2$ are two cuttimes, then

$$a(g_2) - a(g_1) \leq a'(g_2) - a'(g_1),$$

in particular, beads are smaller when measured by a' then when measured by a. Let $E \subset \mathbb{H}\backslash A$ a generic Borel subset, and let $\mu'(E)$ denote $\mu(E)$ measured for the f-mapping of the process B. Since a.s. $\mu(E)$ can be computed as the $\varepsilon \to 0$ limit of the rescaled number of beads of size at least ε starting at a cutpoint in E, it follows that we have $\mu'(f(E)) \leq \mu(E)$. By the restriction property,

$$\overline{\mu}(f(E)) = \mathbf{E}\left(\mu'(f(E))|B \text{ avoids } A\right) \leq \mathbf{E}(\mu(E)|B \text{ avoids } A),$$

and therefore

$$\overline{\mu}(f(E)) \leq \mathbf{E}(\mu(E))/\mathbf{P}(B \text{ avoids } A) = \overline{\mu}(E)(f'(0)f'(\infty))^{-1}.$$

Let $D_r(z)$ denote the open disk of radius r about z. Let K be a compact subset of $\mathbb{H}$. If $z \in K$ and w is sufficiently close to 0, then there exists a subdomain map f so that $f(z) = w$. Consider the set of points w for which f exists for all $z \in K$; this set contains a rescaled version sK of K. By compactness, we may choose subset maps for each $z \in K, sw \in sK$ so that

$$\begin{aligned} c_0 &\leq f'(0)f'(\infty), \\ D_{sr}(sw) &\subseteq f(D_{c_2 r}(z)) \qquad \text{for all } r < c_1 \end{aligned}$$

with constants c_0, c_1, c_2 depending on K only. Then

$$\begin{aligned} s^{2\alpha}\overline{\mu}(D_r(w)) &= \overline{\mu}(D_{sr}(sw)) \\ &\leq \overline{\mu}(f(D_{c_2 r}(z))) \\ &\leq (f'(0)f'(\infty))^{-1}\overline{\mu}(D_{c_2 r}(z)) \\ &\leq c_0^{-1}\overline{\mu}(D_{c_2 r}(z)). \end{aligned}$$

This uniform bound implies the claim by standard arguments about absolute continuity. □

The following are needed for the proof of

Theorem 5. *The Brownian bead index* α *equals* $1/2$.

Let $D_r = D_r(i)$ denote the open disk of radius r about i. Let τ be the (possibly infinite) first hitting time of D_r for the process $B \sim \mathsf{BE}$. Let $B^{(1)}$ denote the path image $B[0, \tau]$. Recall the definition of the capacity $\text{cap}_0(D, A)$ of A from ∞ in D from Section 4, and use the shorthand $\text{cap}_0(t, A) = \text{cap}_0(\rho B_t, A)$. Let $K_r = \text{cap}_0(\tau, D_r)$, and recall the notation $x_r \asymp y_r$ for the existence of a constant $c > 0$ so that $c^{-1}y_r \leq x_r \leq cy_r$ (here for all small r).

Proposition 28. *As* $r \to 0$ *we have* $\overline{\mu}(D_r) \asymp |\log r|\mathbf{E}K_r^{1+2\alpha}$.

Let $B^{(2)}$ denote the path after the last exit from D_r. Let $Z_r' = \mathbf{P}(B^{(1)} \not\pitchfork B^{(2)} \,|\, \mathcal{F}_\tau)$, where "$\not\pitchfork$" denotes "does not intersect". Let Z_r denote the probability given $\mathcal{F}_\tau$ that the image of an independent process $\mathsf{BE}(\infty, D_r, \mathbb{H})$ does not intersect $B^{(1)}$; recall that this image can be defined via conformal mapping of BE started at a more conventional boundary point.

Lemma 29. *We have* $|\log r|K_r \asymp Z_r \asymp Z_r'$, *with deterministic constants.*

Proof. We will use the time-reversal property of Brownian excursion: if $B \sim \mathsf{BE}(a,z,D)$ and ends at random time τ, then $(B(\tau - t), t \in [0,\tau]) \sim \mathsf{BE}(z,a,D)$. This, as well as conformal invariance, can be used to define the image of $\mathsf{BE}(\infty, z, \mathbb{H})$ (together with a time-parameterization starting at $-\infty$, but we won't need this).

The quantities in question are given by the measures of paths that do not intersect $B^{(1)}$ before hitting D_r under the measures $|\log r|\mathsf{BE}(\infty,\mathbb{R},\mathbb{H})$, $\mathsf{BE}(\infty, D_r, \mathbb{H})$, and $\mathsf{BE}(\infty, B(\tau), \mathbb{H})$, respectively. It follows from the definition of BE that the distribution of these paths up to the hitting time of D_r given the hitting position agree. It is easy to check that the in each case hitting position has a smooth density bounded below and above with respect to uniform measure on ∂D_r. The $|\log r|$ normalizing factor is necessary so that the measure of paths hitting D_r is bounded below and above by constants as $r \to 0$. □

Proof of Theorem 5. By the Lemma 27, as $r \to 0$ we have

$$\overline{\mu}(D_r) \asymp r^2.$$

We have

$$\mathbf{E}K_r^\gamma \asymp |\log r|^{-\gamma}\mathbf{E}Z_r^\gamma = r^{\xi(1,\gamma)+o(1)},$$

where the first approximation follows from Lemma 29, and the second is one definition of the intersection exponent. More precisely, we should consider the analogue of Z_r for $B^{(1)'}$, which has distribution $\mathsf{BE}(0, D_r, \mathbb{H})$, but this is absolutely continuous with density bounded above and below with respect to the distribution of $B^{(1)}$ given that it hits D_r (see the proof of Lemma 29).

Thus by Proposition 28 we get $r^{\xi(1,1+2\alpha)+o(1)} = r^2$, and the theorem follows by the monotonicity of $\xi(1,\cdot)$ and the known value $\xi(1,2) = 2$ (see Lawler (1995) for these properties of ξ). □

Proof of Proposition 28.

Upper bound. We write

$$\mathbf{E}(\mu(D_r)\,|\,\mathcal{F}_\tau) = \mathbf{E}(\mu(D_r)\,|\,B^{(1)} \not\pitchfork B^{(2)}, \mathcal{F}_\tau)\mathbf{P}(B^{(1)} \not\pitchfork B^{(2)}\,|\,\mathcal{F}_\tau).$$

Let f denote the conformal shift at the first cuttime g after time τ. We have

$$K_r = \mathrm{cap}_0(\tau, D_r) \geq \mathrm{cap}_0(g, D_r) = \mathrm{cap}_0(f(D_r)).$$

Here, by slight abuse of notation, $f(D_r)$ denotes the image of the part of D_r on which f is defined. By (16) the above implies that $f(D_r) \subset \mathbb{R} \times (0, cK_r]$. By scaling and Lemma 26 we have $\overline{\mu}(\mathbb{R} \times (0, K_r/2]) = cK_r^{2\alpha}$. Thus by the cuttime Markov property (Proposition 14) we get

$$\mathbf{E}(\mu(D_r)\,|\,B^{(1)}\not\pitchfork B^{(2)}, \mathcal{F}_\tau) = \mathbf{E}(\overline{\mu}(f(D_r))\,|\,\mathcal{F}_\tau) \le cK_r^{2\alpha}.$$

Here and in the sequel c denotes a constant whose value may change by line to line. By Lemma 29

$$\mathbf{E}(\mu(D_r)\,|\,\mathcal{F}_\tau) \le cZ_r' K_r^{2\alpha} \le c|\log r| K_r^{1+2\alpha},$$

which implies the upper bound in Proposition 28.

Lower bound. Let A_1 denote the event that $B[0,\tau]$ does not intersect the set $\{w \in D_{2r} : \arg(w-i) \in (\pi/4, 7\pi/4)\}$. It is easy to check (see, for example Lawler et al. (2002)) that for all $\gamma \in [1,3]$ and $r < 1/2$ we have

$$\mathbf{E}(Z_r^\gamma;\, A_1) \ge c\mathbf{E}Z_r^\gamma. \tag{25}$$

Consider the event

$$A = \{B(g) \in D_r,\ B[0,g]\not\pitchfork D_{r/2},\ \mathrm{cap}_0(g, D_{r/32}) \ge cK_r\}.$$

This implies $B^{(1)}\not\pitchfork B^{(2)}$, and it is easy to check that there is an absolute constant c_1 so that

$$\mathbf{P}(A\,|\,B^{(1)}\not\pitchfork B^{(2)}, \mathcal{F}_\tau) \ge c_1 \mathbf{1}_{A_1}.$$

We have

$$\begin{aligned} \mathbf{E}(\mu(D_r)\,|\,\mathcal{F}_\tau) &\ge \mathbf{E}(\mu(D_r);\, A\,|\,\mathcal{F}_\tau)\mathbf{1}_{A_1} \\ &\ge \mathbf{E}(\mu(D_r)\,|\,A, \mathcal{F}_\tau)\mathbf{P}(A\,|\,B^{(1)}\not\pitchfork B^{(2)}, \mathcal{F}_\tau)\mathbf{P}(B^{(1)}\not\pitchfork B^{(2)}\,|\,\mathcal{F}_\tau)\mathbf{1}_{A_1}. \end{aligned}$$

By the cuttime Markov property (Proposition 14) we get

$$\mathbf{E}(\mu(D_r)\,|\,A, \mathcal{F}_\tau) = \mathbf{E}(\overline{\mu}(f(D_r))\,|\,A, \mathcal{F}_\tau).$$

The event A implies that the domain of f contains $D_{r/2}$, and therefore by the Köbe quarter theorem $f(D_{r/2})$ contains a ball D' of radius $|f'(i)|r/8$ centered at $f(i)$. By the same theorem applied to f^{-1}, we get that $f^{-1}(D')$ contains $D_{r/32}$. By monotonicity and conformal invariance of harmonic measure, we get

$$\mathrm{diam}(D') \ge c\mathrm{cap}_0(D') = c\mathrm{cap}_0(g, f^{-1}(D')) \ge c\mathrm{cap}_0(g, D_{r/32}) \ge c' K_r,$$

where the last inequality is an assumption in A. Since $f(\partial D_r)$ contains an curve K that separates 0 and D' from ∞ in $\mathbb{H}$, we have

$$\mathrm{dist}(D', 0) \leq \mathrm{diam}(K) \leq c\,\mathrm{cap}_0(K) \leq c\,\mathrm{cap}_0(f(D_r)) \leq cK_r.$$

When $K_r = 1$, this implies that $\overline{\mu}(D')$ is bounded below by a constant. The scaling property then implies that in general $\overline{\mu}(D') \geq cK_r^{2\alpha}$. Putting all the above together and using Lemma 29 we get

$$\mathbf{E}(\mu(D_r) \,|\, \mathcal{F}_\tau) \geq cZ_r' K_r^{2\alpha} \mathbf{1}_{A_1} \geq c|\log r| K_r^{1+2\alpha} \mathbf{1}_{A_1}$$

and the bound follows from (25). □

Open questions and conjectures

Question 30. The parameter a_0 in the conformal shift (20) defines the "horizontal" location of a cutpoint. It follows from our proofs that the process $((a(\lambda), a_0(\lambda)),\ \lambda \geq 0)$ is a 2-dimensional Lévy process stable under scaling with exponents 1/2 and 1 in the two respective coordinates. In particular $(a_0(\lambda),\ \lambda > 0)$ is a Cauchy process. What is the joint distribution of the two processes? (This question may be better considered in the SLE_6 framework.)

Question 31. Is it possible to interpret bead local time as a local time at zero of some process? This may be better answered by considering a version of SLE_6.

Conjecture 32. Consider the σ-finite bead process conditioned to survive up to distance r (or time r^2, or imaginary part r), and let $r \to \infty$. The limiting process exists and has the restriction property. The exponent (Werner, 2003) should equal 2 (see Lawler et al. (2003) for definitions).

Question 33. The bead time λ defines a random measure on the half-plane. The closed support of this measure is the set of cutpoints. Is it possible to derive the Hausdorff dimension of the set of cutpoints using this measure, or more generally, using beads? This would give a new, conceptual proof for the value of the intersection exponent $\xi(1, 1) = 5/4$.

Acknowledgments. The author thanks Jim Pitman for stimulating remarks and recommending the reference Greenwood and Pitman (1980), which can also be used to construct bead local time. He also thanks Oded Schramm and Wendelin Werner for inspiring discussions and a conscientious referee for several important remarks and corrections.

References

V. Beffara (2003). Hausdorff Dimensions for SLE_6. arXiv:math.PR/0204208. Preprint.

J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

K. Burdzy (1989). Cut points on Brownian paths. *Ann. Probab.*, 17(3):1012–1036.

K. L. Chung (1984). The lifetime of conditional Brownian motion in the plane. *Ann. Inst. H. Poincaré Probab. Statist.*, 20(4):349–351.

M. Cranston and T. R. McConnell (1983). The lifetime of conditioned Brownian motion. *Z. Wahrsch. Verw. Gebiete*, 65(1):1–11.

J. Dubedat (2003). $SLE(\kappa, \rho)$ martingales and duality. arXiv:math.PR/0303128. Preprint.

P. Greenwood and J. Pitman (1980). Construction of local time and Poisson point processes from nested arrays. *J. London Math. Soc. (2)*, 22(1):182–192.

J. F. C. Kingman (1973). Homecomings of Markov processes. *Advances in Appl. Probability*, 5:66–102.

G. F. Lawler (1995). Nonintersecting planar Brownian motions. *Math. Phys. Electron. J.*, 1:Paper 4, approx. 35 pp.

G. F. Lawler (1996). Hausdorff dimension of cut points for Brownian motion. *Electron. J. Probab.*, 1:no. 2, approx. 20 pp.

G. F. Lawler (2001). An introduction to the stochastic Loewner evolution. Preprint.

G. F. Lawler (2003). Book in preparation.

G. F. Lawler, O. Schramm, and W. Werner. Sharp estimates for Brownian non-intersection probabilities. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 113–131. Birkhäuser Boston, Boston, MA, 2002.

G. F. Lawler, O. Schramm, and W. Werner (2003). Conformal restriction: the chordal case. arXiv:math.PR/0209343. Preprint.

G. F. Lawler and W. Werner (2003). The Brownian loop soup. arXiv:math.PR/0304419. Preprint.

P. Lévy (1940). Le mouvement brownien plan. *Amer. J. Math.*, 62:487–550.

C. Pommerenke. *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.

W. Werner (2003). Girsanov's transformation for SLE(κ,ρ) processes, intersection exponents and hiding exponents. arXiv:math.PR/0302115. Preprint.