Research Statement

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My research is in the field of Probability and its applications, particularly to Combinatorics. I am interested in the large scale behaviour of discrete models and in the interface between the discrete models and their (continuous) scaling limits. I have worked on various models of random graphs, including random planar maps and percolation on finite graphs, and have looked at their geometric and topological properties. Another part of my research involves dynamical systems, including cellular automata and various exclusion processes and their applications. Finally, I am also interested in problems concerning various random walks.

Random Graphs and their Geometry

Planar Graphs

Interest in planar maps arises from several different fields. For combinatorialists, the four colour problem has been a driving force behind the study of planar graphs and maps. This has led to a line of research starting with Tutte's work in the 1960's on enumeration of planar maps. Physically, summation over planar maps is seen as a discretized two dimensional analogue of Feynman integrals over paths. This has led physicists to study planar maps under the heading of "2-dimensional quantum gravity". Geometrically, planar maps suggest many questions concerning the geometry of typical surfaces, not restricted by an a-priori Euclidean or hyperbolic structure.

With Oded Schramm, we constructed a new and natural measure supported on infinite planar maps [14], arising as a limit of finite maps. In [3], I have developed an exploration process for planar maps. By analyzing the evolution of the exploration process, I was able to show that in random planar maps the ball of radius R has volume of order R^4. This differentiates the typical planar map from both the Euclidean geometry and from hyperbolic geometries, and confirms physical heuristics that the scaling limit of the random planar map has Hausdorff dimension 4.

The exploration process is especially (though not exclusively) suited for the study of planar triangulations — maps where all faces are triangles. This is notable in light of the difficulty of applying other techniques to study maps with odd face sizes (or their duals). In this setting I have used it to study percolation on random maps, resulting in an analogue for Cardy's formula for the scaling limit of percolation on random planar maps [4].

A long term research goal is to understand the conformal structure of the scaling limit of these maps. This is believed to be some measure on surfaces which under weak conditions does not depend on the local structure of the objects being scaled (as Brownian motion does not depend on the random walk being scaled). The scaled object is the subject of many heuristic results in the physics literature. There is also a possible connection between the scaling limit and random quasi-conformal mappings.

A second goal is to extend some of the results to planar maps coupled with other statistical physics models such as the Ising model. Here the graph underlying the model is interacting with the model itself. This problem requires developing a better understanding of the effect of boundary conditions of the model on the geometry of the graph.

Weak Graph Limits

A theorem of Benjamini and Schramm [15] states (in an equivalent form) that planar graphs with bounded degrees are uniformly recurrent: The time it takes a random walk started at a random vertex to return to its starting point is bounded independently of the graph. (Starting at a uniform vertex is a fundamental point here). With Balazs Szegedy we have been able to prove the conjecture that the condition of planarity can be relaxed. Specifically, we extend this result to any family of graphs with an excluded minor.

This work gives rise to a number of fundamental questions. What is the exact condition for a class of graphs to be uniformly recurrent? Our method works for even larger families of graphs, but there is no exact condition yet. Is there an ergodic theory on limits of graphs with uniform starting points? Such a theory would allow a much better understanding of the possible structure of a graph near a randomly chosen vertex.

Geometry of Percolation Clusters

A different line of research involves the effect of percolation on the geometry of given graphs. Percolation on finite graphs is an active topic of research. In particular, percolation on the hypercube $\{0,1\}^n$ has been studied extensively, with a fundamental result in [1] establishing the existence of a phase transition when the edge retention rate is $p \approx 1/n$, at which point a giant connected component appears. In [7] with Itai Benjamini, we have studied geometric and topological properties of the giant cluster, and have discovered a second phase transition at $p \approx 1/\sqrt{n}$. This phase transition manifests itself in several ways, e.g. in the phase below $1/\sqrt{n}$ the giant component is not locally connected. This transition is the finite graph analogue of the transition from multiple to a unique infinite cluster observed in percolation on some infinite graphs.

The geometric phase transition has a number of consequences. One interesting outcome of the transition concerns the complexity of routing on the percolated hypercube — a problem that arises in computer science in the study of faulty networks. In [8] we have shown that for $p \gg 1/\sqrt{n}$ there is an efficient routing algorithm, while for $p \ll 1/\sqrt{n}$ we give a fractional exponential lower bound on the complexity of any local routing algorithm. One of the problems I will be considering is to prove a similar bound for non-local routing algorithms using a full oracle.

Such a transition does not occur for all graphs, e.g. there is no comparable transition in Euclidean lattices. A fundamental problem of infinite percolation theory is to understand the conditions under which a non-uniqueness phase exists. My main goal in this line of research is to solve the analogue problem in the setting of finite graphs: when can a giant component appear but not be locally connected. Expanders are a key example of graphs where such behaviour seems likely.

Invasion Percolation

Invasion percolation is a growth model for graphs in a random environment. Each edge of a graph is assigned a random resistance. Starting with a single vertex the process repeatedly invades the edge of least resistance accessible from any of the previously reached vertices. This models, for example, the percolation of pressurized liquids through non-porous materials. This model is closely related to the incipient infinite cluster.

In a paper with Goodman, den Hollander and Slade, we analyzed the scaling behaviour of the invasion percolation cluster on a regular tree as a precursor to possible work on $\mathbb{Z}^d$. We give a combinatorial description of the invasion percolation cluster. We then use this description to determine the scaling laws for the size of the cluster at a given height in the tree and for its size up to a given height. Using the description of the cluster, we can also show that while locally the invasion percolation cluster and the incipient infinite cluster are (asymptotically) the same, they differ globally. With Goodman and Merle, we analyzed the asymptotics and scaling limits of invasion percolation on regular trees.

Some further questions involve extending the results to more general settings, from Galton-Watson random trees, to hyperbolic graphs, and as a long term project — which still offers many difficulties — to the lattice $\mathbb{Z}^d$.

Spectral theory of graph limits

Another work concerns the non-backtracking random walk, which is not allowed to retrace its last step, and non-backtracking paths. The generating operator for non-backtracking walks arises in Friedman's research on a generalization of Alon's conjecture about the spectrum of the discrete

Laplacian of random graphs. This operator also appears in lower bounds for the size of graphs with large girth. Its spectrum is the inverse of the poles of the Ihara ζ-function — a graph theoretic analogue of Riemann's ζ-function.

With Friedman and Hoory, we have been able to give an algebraic description of the spectrum for certain infinite trees (universal covers of finite graphs, to be precise). The spectrum is surprisingly complex even for very simple base graphs. An important question we hope to resolve is to find conditions under which that the spectrum of the operator corresponding to a large lifting of a finite graph approximates the spectrum for the operator for the universal cover of the graph. This is true for self-adjoint operators, and appears to hold for a much larger class of operators.

Constrained Optimization

A second area of research involves the effect of various constraints on classic optimization problems. For example, it is known that if edges of a complete graph on n vertices are given independent weights, uniform on $[0, 1]$, then the minimal total weight of a spanning tree (MST) tends to $\zeta(3)$ as $n \to \infty$ [19]

With Flaxman, Wilson and Zecchina, [10] we consider the MST when the diameter is restricted to k (with no constraints the diameter is of order $N^{2/3}$. It turns out that if $k < (\log_2 \log n)/2$ then the total weight grows very quickly, whereas if $k > (\log_2 \log n)/2$ then it stays close to $\zeta(3)$.

Sequential algorithms arise when a set of decisions need to be made sequentially as information is being revealed, with no possibility of reversing a previous decision. E.g. a seller must accept or reject an offer without knowledge of future offers. The average outcome of such algorithms is necessarily worse than the optimal decisions based on the full information. The general problems are finding optimal algorithms and quantifying the degradation in the mean result. Problems that are classically "easy" can become challenging in this setting.

With Aldous and Berestycki, [2] we are studying the optimal total cost, which turns out to be some constat strictly greater than $\zeta(3)$. Our methods use coallescent processes to bound the involved costs.

Interacting particle systems and applications

Interacting particle systems, and exclusion processes in particular, are a third area in my research.

Exclusion Processes

In the totally asymmetric exclusion process (TASEP) with several particle classes, particles on a line randomly attempt to move to the right into empty spaces or into spaces occupied by particles of a lower class. Motivated by some conjectures regarding the stationary measures for these processes, I was led to study the stationary measures for the TASEP with 2 particle classes, and found a surprising combinatorial representation of the stationary measures [5]. This representation sheds new light on some known results concerning the process, both with second class particles (e.g. the renewal at second class particles) as well as for the exclusion viewed from a single second class particle (e.g. convergence to product measures).

The description I gave also gives a new explanation for the matrix ansatz construction of the stationary distribution, Finally, the description highlights a fundamental relation between the TASEP and queues. This relation is different from the standard representation of the TASEP as a series of queues in that the space for the TASEP is the time for the queue, and the classes of particles correspond to the queues. The queueing theory interpretation of my representation has been used in [18] to generalize my construction to an arbitrary number of classes, leading to a proof of some of the original conjectures.

There are several related problems I am still interested in. One involves splitting a class of particles into two classes, similar to the separation of second class particles from first class ones. When the split class in not the first class, there is still no satisfactory description of the resulting distribution. A related problem concerns the result of [17] that with suitable initial conditions, a single second class particle chooses a speed uniformly in $[-1, 1]$ and moves linearly at that speed. I hope to generalize this result to obtain the joint distribution of several second class particles.

Sorting Networks

Another project concerns random sorting networks. Here, n numbers are initially in order, and adjacent pairs are exchanged until the order is reversed (swaps occur one at a time). We are interested in minimal sorting networks, which are of length $\binom{n}{2}$. Sorting networks can equivalently be defined as minimal representations of $(n, n-1, \ldots, 1) \in S_n$ using the adjacent transpositions as generators, or geodesic paths in the corresponding Cayley graph of S_n. Of primary interest is the uniform sorting network which is chosen uniformly from among all minimal sorting networks.

A fundamental result by Stanley is that the number of sorting networks of size n is equal to the number of Young tableaux of shape $(n, n-1, \ldots, 1)$, which is $\binom{n}{2}! \cdot \prod(2i-1)^{n-i}$. Edelman and Greene [16] gave an explicit bijection between sorting networks and Young tableaux. With Holroyd, Romik and Virag [12], we are interested in the paths that numbers take in the uniform sorting network, and study both the distribution of individual paths and their joint distribution. In particular, there is strong evidence that in the scaling limit individual paths are close to sine curves with random phase and amplitude. This is related to the random speed of the second class particle in a TASEP, in that the randomness in the large scale behaviour of the path up to time $\binom{n}{2}$ is determined very early.

There is a natural bijection (by Edelman-Greene) between sorting networks and Young Tableaux of a certain shape. We show that some properties of sorting networks such as the distribution of transposition locations can be derived from this bijection. We also show that many properties follow from the conjecture that with a natural embedding of S_n in $\mathbb{R}^n$, most geodesic paths (they are not unique) lie close to great Euclidean circles. With Holroyd and Romik [11] we study some other distributions on sorting networks, more closely related to the TASEP, and with Holroyd and Virag [13] the local structure of the sorting networks.

The major open problems that I hope to solve in this area are to show that the particle trajectories are indeed sine curves with random amplitudes, and to understand the correlations between them, which show non-trivial behaviour.

Random walks

As a final aspect of my research, I'll describe some of my work on various random walks and related models.

Diffusion limited aggregation

Diffusion limited aggregation (DLA) is a notoriously difficult model to analyze, despite its simple definition and natural motivation. It models several growth processes, among others of crystal growth in a solution and some aspects of oil/water interface in a thin layer. In the standard DLA model an aggregate is grown by particles that perform a random walk in two dimensions, and freeze once they hit the aggregate. With hundreds of papers written about it, the only significant rigorous result is due to Kesten.

With Amir, Benjamini, and Kozma, [6] we are studying a version of DLA in 1 dimension. We replace the two dimensional random walk by a random walk in 1 dimension with unbounded steps. Suitable distributions for the one dimensional random walk result in models that asymptotically approximate the standard DLA model. As for DLA in higher dimensions, the main focus of interest is the growth rate of the aggregate, related to its dimension. We consider the dependence of the growth rate on the step distribution for the generating random walk, and in particular on the distribution's tail. We show that the growth exhibits three phase transitions, when the random walk steps have finite $\frac{1}{2}$th moment, finite variance, and finite third moments.

Some further research into the structure of the resulting aggregate is in preliminary stages.

Spatial Coalescents

Kingman's coalescence [20] is a fundamental model for the ancestral trees of a present population, which can model family trees or evolution of species. A growing avenue of research involves the effect of a spatial structure on the dynamics. Particles (individuals) migrate in a given space and interact only with others in their current location.

With Berestycki, Hammond and Limic, [9] we consider a spatial version of Kingman's coalescence. We prove that if one traces the ancestry of N individuals for some fixed time t, the number of disjoint ancestral trees grows extremely slowly: as $\log^*(N)$ (where $\log^*(x) = k$ if the k'th log of x is in $(0, 1]$). We can also calculate the rate of growth for more general coalescence models.

One of the major drawbacks of Kingman's coalescent is that it only describes ancestry along a single parent (e.g. mitochondrial DNA), and does not describe recombinant DNA where an individual has two parents. I hope to extend some of the very detailed theory of Kingman's coalescence to this more general model.

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi. Largest random component of a k-cube. *Combinatorica*, 2(1):1–7, 1982.

[2] D. Aldous, O. Angel, and N. Berestycki. Sequential minimal spanning trees. ~13 pages, in preparation.

[3] O. Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003.

[4] O. Angel. Scaling of percolation on infinite planar maps. *preprint*, 2005. arXiv:math/0501006.

[5] O. Angel. The stationary measure of a 2-type totally asymmetric exclusion process. *J. Combin. Theory Ser. A*, 113(4):625–635, 2006.

[6] O. Angel, G. Amir, G. Kozma, and I. Benjamini. 1 dimensional DLA. ~40 pages, in preparation.

[7] O. Angel and I. Benjamini. A phase transition for the metric distortion of percolation on the hypercube. *Combinatorica*, 2003. arXiv:math.PR/0306355. 11 pages, to appear.

[8] O. Angel, I. Benjamini, E. Ofek, and U. Wieder. Routing complexity of faulty networks. In *PODC*, pages 209–217, 2005. (to appear, Rand. Struct. Algorithms.

[9] O. Angel, N. Berestycki, A. Hammond, and V. Limic. Global divergence of spatial coalescents. ~44 pages, in preparation.

[10] O. Angel, A. Flaxman, D. Wilson, and R. Zecchina. Diameter constrained minimal spanning trees. ~10 pages, in preparation.

[11] O. Angel, A. Holroyd, and D. Romik. The circected permutahedron random walk. in preparation.

[12] O. Angel, A. Holroyd, D. Romik, and B. Virag. Random sorting networks. *Adv. in Math.*, 215(2):839–868, 2007.

[13] O. Angel, A. Holroyd, and B. Virag. The weak limit of boundary young tableaux. in preparation.

[14] O. Angel and O. Schramm. Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241(2-3):191–213, 2003.

[15] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 1–13, 2001.

[16] P. Edelman and C. Greene. Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the weak Bruhat order of S_n. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 155–162. Amer. Math. Soc., Providence, RI, 1984.

[17] P. Ferrari and C. Kipnis. Second class particles in the rarefaction fan. *Ann. Inst. H. Poincaré Probab. Statist.*, 31(1):143–154, 1995.

[18] P. Ferrari and J. Martin. Multiclass processes, dual points and M/M/1 queues. *Markov Process. Related Fields*, 12:175–201, 2006.

[19] A. M. Frieze. On the value of a random minimum spanning tree problem. *Discrete Appl. Math.*, 10(1):47–56, 1985.

[20] J. F. C. Kingman. The coalescent. *Stochastic Process. Appl.*, 13(3):235–248, 1982.