MAT 137Y - Practice problems
Unit 14 - Power series and Taylor series

1. Find the interval of convergence of the following power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} n(x+1)^n$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n\, 3^{2n+1}}{\ln n}(x-1)^n$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{n!}x^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} n!(x+1)^n$

2. Compute the Maclaurin series of the following functions by performing algebraic manipulations to the four main Maclaurin series. Indicate the domain where the expansion is valid.

   (a) $f(x) = \dfrac{1}{1+x}$

   (b) $f(x) = \dfrac{x}{1-x^2}$

   (c) $f(x) = x^2 e^x$

   (d) $f(x) = \dfrac{e^x + e^{-x}}{2}$

   (e) $f(x) = \cos\left(3x^4\right)$

   (f) $f(x) = \dfrac{1}{2+x^2}$

3. Compute the Maclaurin series of the following functions using differentiation or integration of other known series. Indicate the domain where the expansion is valid.

   (a) $f(x) = \ln(1+x)$

   (b) $f(x) = \arctan x$

4. Consider the function $F(x) = \displaystyle\int_0^x e^{-t^4} dt$. It is impossible to find an "elementary antiderivative" for the function $f(t) = e^{-t^4}$, so we use series instead to understand this function.

   (a) Obtain the Taylor series of $f(t) = e^{-t^4}$ around $t = 0$.

   (b) Use the previous answer to represent the function $F(x) = \displaystyle\int_0^x e^{-t^4} dt$ as a power series.

   (c) Estimate $\displaystyle\int_0^1 e^{-t^4} dt$ with an error smaller than 0.001.
   Hint: Notice the series is alternating.

5. (a) Let $f(x) = e^{1/x}$. Use the Maclaurin series of $g(x) = e^x$ to write f as a "power series with negatives exponents". For which values of x is $f(x)$ actually equal to this series?

   (b) Let $h(x) = xe^{1/x}$. Use your answer to Question 5a (write down the first few terms to see what it looks like) to find the slant asymptote of the function h.

   Hint: This is very, very short. Very.

6. Let $f(x) = e^{x^{10}} + e^{-x^{10}} + 2\cos(x^{10}) - 4$. Find the first non-zero derivative of f at 0. In other words, find the smallest value of $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$ and the value of that derivative.

7. Use Taylor series to compute the following limits

(a) $\displaystyle \lim_{x \to 0} \frac{\sin x - x\cos x - \sin\frac{x^3}{3}}{x^5}$

(b) $\displaystyle \lim_{x \to 0} \frac{x\,(e^x - 1)^5 \ln(1+x)}{(\sin x - x)\,(\cos x - 1)^2}$

8. Compute the value of the following series:

(a) $\displaystyle \sum_{n=1}^{\infty} n x^n$

(b) $\displaystyle \sum_{n=0}^{\infty} \frac{n^2}{2^n}$

(c) $\displaystyle \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

(d) $\displaystyle \sum_{n=0}^{\infty} \frac{(n+1)}{n!}\, x^n$

(e) $\displaystyle \sum_{n=0}^{\infty} \frac{x^n}{(n+3)\, n!}$

(f) $\displaystyle \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{(2n+1)!}\, x^{2n+1}$

(g) $\displaystyle \sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)}$

(h) $\displaystyle \sum_{n=2}^{\infty} \frac{(2n+3)(2n+1)\,2^n}{n!}$

9. In addition to the four main Maclaurin series you have learned, many books include a fifth: the binomial series. You are going to learn about it in this question.

   Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$.

   (a) If $\alpha \in \mathbb{N}$, the function f has a very boring Taylor series. Why?

   (b) From now on assume $\alpha \notin \mathbb{N}$. Find a formula for $f^{(n)}(x)$ and prove it.

   Suggestion: Compute a few terms, guess the pattern, then prove it by induction.

   (c) Write down an explicit formula for the Maclaurin series of $f(x)$. Let us call this series $S(x)$.

   (d) Calculate the radius of convergence of $S(x)$.

   Note: It is possible to prove that $f(x) = S(x)$ inside the interval of convergence, but it requires other versions of the Remainder Theorem. For now, just accept this without proof.

10. Once you have the binomial series, you can obtain more!

   (a) Obtain the Maclaurin series for $g(x) = \arcsin x$. In which domain can you be certain that arcsin is equal to its Maclaurin series?

   Hint: What is $g'(x)$? First, use the binomial series with $\alpha = -1/2$ to write the Maclaurin series for $g'(x)$ and then integrate.

   (b) Calculate $g^{(137)}(0)$.

   Hint: You do not need to take any derivatives now. This should be quick.

   (c) Prove that $\displaystyle \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2\,(2n+1)\,8^n} = \frac{\pi}{2\sqrt{2}}$

11. Use Taylor series and Lagrange Remainder Theorem to estimate $\ln 0.9$ with an error smaller than 0.001.

12. This question is an example of an application of Taylor series to physics, but you do not need to know any physics to solve it.

    Charged particles create a property in space called *electric potential*. If we have a particle with charge q at a point P, it will create an electric potential V at a point Q given by the equation

    $$V = \frac{kq}{r}$$

    where k is a constant and r is the distance between P and Q. We say that the electric potential created by one particle is inversely proportional to the distance.

    (a) Let's say that we have a charge with value q at the point $x = a$ in the x-axis and a charge with value $-q$ at the point with $x = -a$. (We call this a *dipole*.) We want to study how the total electric potential (that is, the sum of the electric potentials created by both particles) depends on the distance at points very far away from both charges. The total electric potential at a point $x > a$ in the x-axis will be

    $$V = \frac{kq}{x-a} - \frac{kq}{x+a}. \tag{1}$$

    Since we are looking at points very far away from both charges, we may assume that x is much bigger than a. Let us call $u = \frac{a}{x}$. Then the quantity u is very small.

    Express Equation 1 in terms of k, q, x, and u (but not a). Then write it as a Taylor series using u as the variable.

    Hint: All you need is the geometric series, which you already know. You do not need to take any derivatives.

    (b) Since u is very small, it makes sense to keep only the first non-zero term of the Taylor series you obtained in Question 12a. Do so. If you do this correctly, you will have proven that the potential created by these two charges together is directly proportional to a and inversely proportional to the square of the distance.

    (c) This time assume that we have a charge with value $-q$ at $x = a$, a charge with value $2q$ at $x = 0$ and a charge with value $-q$ at $x = -a$. Then the total electric potential created by these charges is

    $$V = -\frac{kq}{x-a} + \frac{2kq}{x} - \frac{kq}{x+a}.$$

    Do a calculation similar to the above to answer the following question: For values of x far away from these charges, the total electric potential is inversely proportional to which power of the distance?

Bonus questions: Proof of Lagrange Remainder Theorem

In the videos you learned:

Lagrange Remainder Theorem:

- Let I be an open interval. Let $a \in I$. Let $n \in \mathbb{N}$.
- Let f be a C^{n+1} function on I.
- Let $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ be the n-th Taylor polynomial for f at a.
- Let $R_n(x) = f(x) - P_n(x)$ be the n-th Taylor remainder.
- Let $x \in I$ such that $x \neq a$

Then there exists ξ between a and x such that

$$R_n(x) \;=\; \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-a)^{n+1}$$

Now you can prove it.

13. (a) Write the statement of Lagrange's Remainder Theorem for $n = 0$. Convince yourself that you have already proven it!

    Hint: It is MVT in disguise.

    (b) Review the Generalized Rolle's Theorem. (See Question 8 on Practice Problems for Unit 5. We called it the "N-th Rolle Theorem there.) You will need to use it as a lemma.

    (c) Here is a sketch of the proof of Lagrange Remainder Theorem for $n = 1$.

    Fix $b \in I$ such that $b > a$. I want to define a function h by the equation

    $$h(t) \;=\; f(t) - P_1(t) - M(t-a)^2$$

    for some $M \in \mathbb{R}$ that I will determine later. I am using t as the variable rather than x because we will give a different use to x later. Notice that $h(a) = h'(a) = 0$. Choose a value of M that will make $h(b) = 0$. Once you have that value of M, use the 2nd Rolle's Theorem on h on $[a, b]$. Look at the conclusion.

    Now convince yourself that you have proven Lagrange's Remainder Theorem for $n = 1$.

    (d) Use a similar idea to prove Lagrange's Remainder Theorem for arbitrary n.

Some answers and hints

1. (a) $(-2, 0)$

   (b) $\left(\frac{8}{9}, \frac{10}{9}\right]$

   (c) $(-\infty, \infty)$

   (d) $\{-1\}$

2. (a) $\dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \ldots$ for $|x| < 1$.

   (b) $\dfrac{x}{1-x^2} = \sum_{n=0}^{\infty} x^{2n+1} = x + x^3 + x^5 + x^7 + x^9 + \ldots$ for $|x| < 1$.

   (c) $x^2 e^x = \sum_{n=0}^{\infty} \dfrac{x^{n+2}}{n!} = x^2 + \dfrac{x^3}{1!} + \dfrac{x^4}{2!} + \dfrac{x^5}{3!} + \dfrac{x^6}{4!} + \ldots$ for all x.

   (d) $\dfrac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots$ for all x.

   (e) $\cos\left(3x^4\right) = \sum_{n=0}^{\infty}(-1)^n \dfrac{3^{2n}x^{8n}}{(2n)!} = 1 - \dfrac{3^2x^8}{2!} + \dfrac{3^4x^{16}}{4!} - \dfrac{3^6x^{24}}{6!} + \dfrac{3^8x^{32}}{8!} - \ldots$ for all x.

   (f) $\dfrac{1}{2+x^2} = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{2^{n+1}} = \dfrac{1}{2} - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^3} - \dfrac{x^6}{2^4} + \dfrac{x^8}{2^5} - \ldots$ for $|x| < \sqrt{2}$

3. (a) $\ln(1+x)$ is an antiderivative of $\dfrac{1}{1+x}$

   $$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

   Valid for $|x| < 1$.

   (b) $\arctan x$ is an antiderivative of $\dfrac{1}{1+x^2}$

   $$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

   Valid for $|x| < 1$.

4. (a) $f(t) = \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n}}{n!}$ for all $t \in \mathbb{R}$

   (b) $F(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n+1}}{(4n+1)(n!)}$ for all $x \in \mathbb{R}$

   (c) $\displaystyle\int_0^1 e^{-t^4}dt \approx \sum_{n=0}^{4} \frac{(-1)^n}{(4n+1)(n!)} \approx 0.845\ldots$

5. (a) $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!\,x^n}$ for all $x \neq 0$.

   (b) $y = x + 1$

6. $n = 40$. $f^{(40)}(0) = \frac{40!}{6}$.

7. (a) $\frac{-1}{30}$ (b) -24

8. (a) $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

   (b) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} = 6$

   (c) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = xe^x$ for all x

   (d) $\sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^n = (x+1)e^x$ for all x

   (e) $\sum_{n=0}^{\infty} \frac{x^n}{(n+3)\,n!} = \frac{(x^2 - 2x + 2)e^x - 2}{x^3}$ for $x \neq 0$

   (f) $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{(2n+1)!} x^{2n+1} = \frac{1}{2} [\sin x + x \cos x]$ for all x.

   (g) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} = \frac{x + (1-x)\ln(1-x)}{x^2}$ for $0 < |x| < 1$.

   Note: The identity is also true for $|x| = 1$ as well, but with the tools we have developed in the course, we can only say it is true for $0 < |x| < 1$ for now. It can also be extended to $x = 0$ with a limit.

   (h) $\sum_{n=2}^{\infty} \frac{(2n+3)(2n+1)\,2^n}{n!} = 43e^2 - 33$.

9. (a) It is a finite sum.

   (b) $f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2) \cdot \ldots \cdot (\alpha - n + 1)\,(1+x)^{\alpha - n}$

   (c) $S(x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1) \cdot \ldots \cdot (\alpha - n + 1)}{n!} x^n$

   (d) The radius of convergence is 1.

10. (a) $\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2\ 4^n\ (2n+1)} x^{2n+1}$ for $|x| < 1$

(b) $g^{(137)}(0) = \dfrac{(136!)^2}{2^{136}\,(68!)^2}$

11. Use the Maclaurin series for $\ln(1+x)$ with $x = -0.1$.

$$\ln(0.9) = -\sum_{n=1}^{\infty} \frac{0.1^n}{n}$$

Using Lagrange Remainder Theorem we can bound the error of the n-th polynomial approximation at $x = -0.1$ as

$$|R_n(-0.1)| \leq \frac{1}{n+1}\left(\frac{1}{9}\right)^{n+1}$$

Taking $n = 2$ works. Thus our estimation is

$$\ln(0.9) \;\approx\; P_2(-0.1) = -0.1 - \frac{0.1^2}{2} = -0.105.$$

12. (a) $V = \dfrac{2kq}{x} \displaystyle\sum_{n=0}^{\infty} u^{2n+1}$

(b) For large x, $\quad V \approx \dfrac{2kqa}{x^2}$

(c) For large x, $\quad V \approx \dfrac{-2kqa^2}{x^3}$