1. Determine whether the following integrals are convergent or divergent directly from the definition of improper integral. If they are convergent, calculate their value.

(a) \[ \int_{-1}^{\infty} \frac{1}{x^2 + 1} \, dx \]
(b) \[ \int_{0}^{1} \ln x \, dx \]
(c) \[ \int_{0}^{\infty} \cos x \, dx \]
(d) \[ \int_{0}^{1} \frac{dx}{x^2} \]
(e) \[ \int_{0}^{1} \frac{dx}{\sqrt{x}} \]
(f) \[ \int_{2}^{\infty} \frac{1}{x^2 - 1} \, dx \]

**Hint:** For Question (1f), write \( \frac{1}{x^2 - 1} = \frac{A}{x-1} + \frac{B}{x+1} \).

2. (a) For which values of \( p > 0 \) is the integral \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) convergent?
(b) For which values of \( p > 0 \) is the integral \( \int_{0}^{1} \frac{1}{x^p} \, dx \) convergent?
(c) Let \( a, b \in \mathbb{R} \). Assume \( a < b \).
For which values of \( p > 0 \) is the integral \( \int_{a}^{b} \frac{1}{(x-a)^p} \, dx \) convergent?

3. Using the Basic Comparison Test and/or the Limit-Comparison Test, determine which ones of the following improper integrals are convergent or divergent. Do not calculate their value.

(a) \[ \int_{1}^{\infty} \frac{\sin x + 2 \cos x + 10}{x^2} \, dx \]
(b) \[ \int_{0}^{\infty} \frac{x - 7}{x^2 + x + 5} \, dx \]
(c) \[ \int_{10}^{\infty} \frac{\sqrt{x - 6}}{3x^2 + 5x + 11} \, dx \]
(d) \[ \int_{0}^{\infty} \frac{\arctan x}{x^{1.1}} \, dx \]
(e) \[ \int_{0}^{1} \frac{\sin x}{x^{4/3}} \, dx \]
(f) \[ \int_{0}^{\infty} e^{-x^2} \, dx \]

4. Let \( a < b \). Let \( f \) be a continuous function on \([a, \infty)\). Prove that the following two statements are equivalent:

- The improper integral \( \int_{a}^{\infty} f(x) \, dx \) is convergent.
- The improper integral \( \int_{b}^{\infty} f(x) \, dx \) is convergent.

Write a formal proof directly from the definition of improper integral as a limit.

**Suggestion:** Use the limit laws. You do not need to get dirty with epsilons.
5. Review the statement of the Limit-Comparison Test (Video 12.9). There are two generalizations of the theorem.

(a) Assume the limit $L$ in the theorem exists and is 0. The full conclusion of the theorem is no longer true but we can still draw some conclusions in some cases. If one of $\int_a^\infty f(x)dx$ or $\int_a^\infty g(x)dx$ is convergent or divergent, can we conclude something about the other? Figure out what the correct conclusions are, and write a proof (imitating the proof in Video 12.10).

(b) Repeat the same question when the limit $L$ is $\infty$.

6. For which values of $a, b \in \mathbb{R}$ are each of the following improper integrals convergent or divergent?

(a) $\int_2^\infty \frac{1}{x^a (\ln x)^b} \, dx$    
(b) $\int_1^2 \frac{1}{x^a (\ln x)^b} \, dx$    
(c) $\int_1^\infty \frac{1}{x^a (\ln x)^b} \, dx$

Note: This is a long question. You will have to break each integral into cases (depending on values of $a$ and $b$). You will likely use BCT, LCT, the definition of improper integral, and substitution at different points. For Question 6b, we suggest studying the case $a = 1$ first.

7. • A type-1 improper integral is an integral of the form $\int_c^\infty f(x)dx$, where $f$ is a continuous, bounded function on $[c, \infty)$.

• A type-2 improper integral is an integral of the form $\int_a^b f(x)dx$, where $f$ is a continuous function on $(a, b]$ (possibly with vertical asymptote $x = a$) or on $[a, b)$ (possibly with vertical asymptote $x = b$).

In the Videos we explicitly wrote the statement (and proof) for BCT and LCT for type-1 improper integrals, but we have also been using them for type-2 improper integrals. Write the statements and proofs.
Some answers and hints

1. (a) Convergent: \(\frac{3\pi}{4}\)  
   (c) Divergent: oscillating  
   (e) Convergent: 2  
   (b) Convergent: \(-1\)  
   (d) Divergent: \(\infty\)  
   (f) Convergent: \(\frac{1}{2}\ln 3\)

2. (a) Convergent iff \(p > 1\)  
   (b) Convergent iff \(p < 1\)  
   (c) Convergent iff \(p < 1\)

3. (a) Convergent  
   (b) Divergent  
   (c) Convergent  
   (d) Convergent  
   (e) Convergent  
   (f) Convergent

4. \[
\int_{a}^{\infty} f(x)dx = \lim_{x \to \infty} F(x) \quad \text{where} \quad F(x) = \int_{a}^{x} f(t)dt
\]
\[
\int_{b}^{\infty} f(x)dx = \lim_{x \to \infty} G(x) \quad \text{where} \quad G(x) = \int_{b}^{x} f(t)dt
\]

Notice that \(G(x) = F(x) + M\) where \(M\) is a fixed number (which number?) Use limit law for sum of functions.

5. Let us call \(P = \int_{a}^{\infty} f(x)dx\) and \(Q = \int_{a}^{\infty} g(x)dx\). Let \(L = \lim_{x \to \infty} \frac{f(x)}{g(x)}\).

   (a) Assume \(L = 0\).
   • If \(P = \infty\) then \(Q = \infty\).
   • If \(Q < \infty\) then \(P < \infty\)

   (b) Assume \(L = \infty\).
   • If \(Q = \infty\) then \(P = \infty\).
   • If \(P < \infty\) then \(Q < \infty\).

We cannot draw any other conclusions.

6. (a) Convergent when \(a > 1\). Convergent when \(a = 1\) and \(b > 1\). Divergent otherwise.
   (b) Convergent when \(b < 1\). Divergent when \(b \geq 1\). The value of \(a\) does not matter.
   (c) Convergent when \(a > 1\) and \(b < 1\). Divergent otherwise.