MAT 137Y - Practice problems
Unit 10 - Applications of the integral

PART A - Volumes

1. Consider the region bounded by the curve $y = 2x - x^2$ and the x-axis. We rotate the region around the x-axis. Calculate the volume of the resulting solid of revolution.

   Answer: $16\pi/15$.

2. Derive a formula for the volume of a cone with base radius R and height h.

   Answer: $\frac{1}{3}\pi R^2 h$.

3. Let R be the region bounded by the curves $y = x^2 + 4$ and $y = 6x - x^2$. Calculate the volume of the solid of revolution obtained when we rotate the region R...

   (a) around the x-axis,

   (b) around the y-axis.

   Answers: $13\pi/3$, π.

4. Consider the region in the first quadrant bounded between the x-axis, the y-axis, and the curve $\sqrt{x} + \sqrt{y} = 1$. Calculate the volume of the solid of revolution obtained by rotating this region around the line $x = 3$. Do this in two different ways:

   (a) ...integrating with respect to x.

   (b) ...integrating with respect to y

   Answer: $14\pi/15$.

PART B - Other applications

You have learned to use integrals to compute areas and volumes, but there are many other applications. Our objective is not to teach you any particular application of the integral (there are so many!), but to make you ready for any applications that your second-year professors will throw at you, no matter how hard. In other words, we do not want you to memorize any one formula; we want you to understand where the formulas come from. In the next problems you can practice *deriving the formulas for various applications that we have not taught you anywhere*. That is the entire point.

This problem set looks long. Do not be intimidated! We have broken each section into short pieces to help you. It is actually pretty short.

The average of a function

The *average* of the numbers $a_1, a_2, \dots, a_N$ is defined as the number $\frac{1}{N}\sum_{i=1}^{N} a_i$.

5. As a warm-up compute the average of the numbers $0, 1, 3, 3, 7, -1, 5$.

   Answer: 18/7.

6. In this problem we want to define what it means to compute the average of a function. First, consider the function g with domain $[0, 5]$, defined as

$$g(x) = \begin{cases} 8 & \text{if } 0 \leq x < 1 \\ 3 & \text{if } 1 \leq x < 3 \\ 2 & \text{if } 3 \leq x < 4 \\ 1 & \text{if } 4 \leq x \leq 5 \end{cases}$$

   We call a function like this one *piecewise-constant*. Draw its graph. Looking at the graph, what should the average of this function be?

   Answer: 17/5.

7. Now consider any continuous function f with domain $[a, b]$. To compute its average, pretend the function is piecewise-constant but with infinitely many pieces which are infinitesimally small. (The sum of all those pieces will be some integral.) Obtain a formula for the average of f.

8. To test your formula, compute the average of the function $f(x) = x^3$ on $[0, 3]$.

   Answer: 27/4.

If you got this far, give yourself a cookie.

Mass density

9. As a warm-up, consider a bar which is 6 meters long with a density of $2kg/m$ (2 kilograms per meter). What is its total mass?

   Answer: $12kg$.

10. Now we have a bar which lies on the x-axis, from the position $x = 1$ to the position $x = 7$ (all values of x are measured in meters). The bar is made of different materials. Between $x = 1$ and $x = 2$, the density of the bar is $2kg/m$; between $x = 2$ and $x = 5$, the density is $3kg/m$; between $x = 5$ and $x = 5.5$, the density is $10kg/m$, and between $x = 5.5$ and $x = 7$, the density is $1kg/m$. What is the total mass of the bar?

    Answer: $17.5kg$.

11. This time we have a bar on the x-axis lying between the positions $x = a$ and $x = b$. The density of the bar at at point x is $\mu(x)$, where μ is a continuous function on $[a, b]$. This means that the density may be different at each point. Obtain a formula for the total mass of the bar using integrals.

    Hint: Break the bar into pieces of "infinitesimally small" width dx. Write the mass of the bar as the "sum" (i.e. integral) of the masses of these microscopic pieces.

12. To test your formula, calculate the total mass of a bar with density function $\mu(x) = 1+2x$ with $1 \leq x \leq 3$.

    Answer: 10.

If you got this far, give yourself a chocolate cookie.

Centre of mass

A point mass is a particle with a certain mass that has size 0 and occupies exactly one point on the x-axis. Let's say we have a collection of point masses along the x-axis: a mass m_1 at position x_1, a mass m_2 at position x_2, ..., and a mass m_n at position x_n. The position of the *centre of mass* of this collection is defined as

$$x_{CM} = \frac{1}{M} \sum_{i=1}^{n} x_i m_i,$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass. One way to interpret this is that we are taking the average of their positions, but weighted according to their masses.

13. Now instead of a collection of point masses we have continuous masses (more realistic). Specifically, like in the previous problem, we have a bar on the x–axis from $x = a$ to $x = b$, whose density at the point x is given by $\mu(x)$. Obtain a formula for the position of the centre of mass of the bar using integrals.

14. To test your formula, calculate the position of the centre of mass of a bar with mass density $\mu(x) = x^3$, $0 \leq x \leq 1$.

    Answer: $\frac{4}{5}$.

You've earned yourself some hot cocoa. Baby, it's cold out there!

Arc length

Let $a < b$. Let f be a differentiable function defined on $[a, b]$. We want to compute the length of the graph of f.

15. As a warm-up, calculate the distance between the points $(1, 1)$ and $(4, 5)$.

    Answer: 5.

16. Let P and Q be the points in the graph of f with x-coordinates a and b respectively. Write an equation for the distance between P and Q. Then use the Mean Value Theorem to prove there exists $c \in (a, b)$ such that this distance is

$$\text{dist}(P, Q) = \sqrt{1 + [f'(c)]^2}\,(b - a).$$

17. The result you obtained in Question 16 is an approximation for the length of the graph of f. (It is the exact value when the graph of f happens to be a straight line, but otherwise only an approximation.) Now use a partition of the interval $[a, b]$ to break the graph into many smaller pieces, and use the same approximation for each piece. The result you obtain (a sum) is a better approximation for the length of the graph of f.

18. Now "break the interval $[a, b]$ into infinitely many pieces which are infinitesimally small". Obtain a formula for the length of the graph of f as an integral.

19. To test your formula, calculate the length of the arc of $y = x^{3/2}$ from $x = 0$ to $x = 1$.

    Answer: $\frac{1}{27}\left[13\sqrt{13} - 8\right]$.

20. *[Harder]* Now you have the tools to prove the formula for the length of a circumference! Do it.

You got to the end. Congratulations! Celebrate with some natto.