MAT 137Y - Practice problems
Unit 8 - The Fundamental Theorem of Calculus

1. Compute these antiderivatives:

   (a) $\int x^3\,dx$

   (b) $\int \sqrt{x}\,dx$

   (c) $\int \frac{1}{x^3}\,dx$

   (d) $\int \left(x^3 - 2x^2 + 7x - 5\right)dx$

   (e) $\int (3x+7)^{10}\,dx$

   (f) $\int \cos x\,dx$

   (g) $\int 3\sin(2x)\,dx$

   (h) $\int 5e^{-2x}\,dx$

   (i) $\int \frac{2}{(7-6x)^4}\,dx$

   (j) $\int \frac{x^3+2x^2}{x}\,dx$

   (k) $\int \sqrt{x}\,(x+1)\,dx$

   (l) $\int \frac{1}{x}\,dx$

   (m) $\int \frac{1}{\sqrt[3]{5-2x}}\,dx$

   (n) $\int \sec^2 x\,dx$

   (o) $\int \tan^2 x\,dx$

   Hint: You can solve all of these by "guess and check". Make an educated guess, try it, and then take modify it if needed.

2. Compute the derivatives of the following functions:

   (a) $F(x) = \int_0^x \frac{1}{t^3+1}dt$

   (b) $F(x) = \int_0^{x^2} \frac{1}{t^3+1}dt$

   (c) $F(x) = \int_{x^2}^0 \sin(\cos t)dt$

   (d) $F(x) = \int_{\sin x}^{\cos x} \ln(t^2+1)dt$

3. Calculate

   (a) $\int_0^1 x^5\,dx$

   (b) $\int_1^3 e^{2x}\,dx$

   (c) $\int_0^\pi \sin x\,dx$

4. Compute the area of the following regions:

   (a) the region bounded by the graph $y = 4x - x^2$ and the x-axis.

   (b) the region bounded by the curves $y = x^3 - 2x + 2$ and $y = x^2 + 2$.
   Hint: These two curves meet at three different points.

   (c) the region between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$
   Hint: Use y as the variable.

5. The graphs of $y = x$, $xy = 1$, and $2x = y + 1$ split the plane into various regions. Only one of those regions is bounded. Compute its area.

6. Let

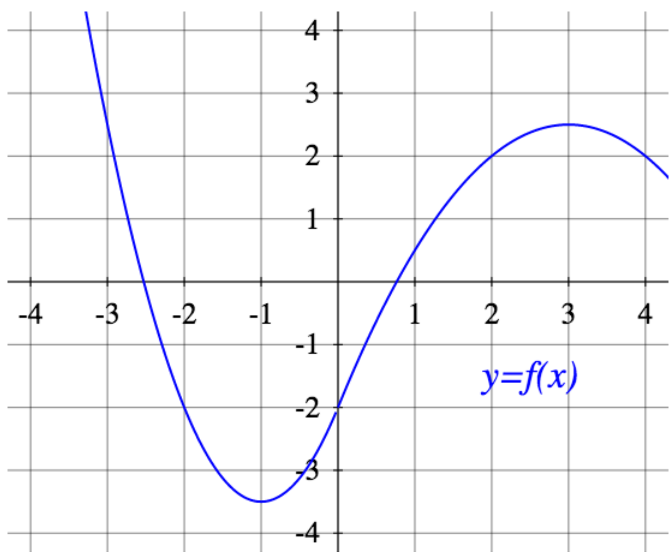
$$H(x) = \int_{x^2-3x}^{x^2+x} \left(1 + \sin t^3\right) dt$$

Find the equation of the line tangent to the graph of $y = H(x)$ at the point with x-coordinate 1.

7. Let $a, b, c, k \in \mathbb{R}$. Compute the following limit

$$\lim_{x \to 0} \frac{\int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds\right] dt}{\cos x - 1}$$

8. Below is the graph of the function f:



Find the local extrema of the function F defined by the equation

$$F(x) = \int_0^x \left[(f(t))^2 - t^2\right] dt.$$

Note: Your answers may be approximate, since you only have a graph and not an equation.

9. Review the statement and proof of Part 1 of the FTC (Videos 8.3 and 8.4). There is a slightly different version of the same theorem that often appears in books, but it is a bit harder to prove:

Theorem [FTC 1']
Let I be an open interval. Let f be a bounded, integrable function on I. Let $a \in I$.
We define a new function F by the equation $F(x) = \displaystyle\int_a^x f(t)dt$.
Let $b \in I$.
IF f is continuous at b THEN F is differentiable at b and $F'(b) = f(b)$.

(a) Compare this theorem with the one in Video 8.3 and make sure you understand the difference.

(b) If you try to imitate the proof from Video 8.4 in this case, it mostly works exactly the same, but there is one important step that is not straightforward. Go through the proof and identify this step.

(c) For each $h > 0$, let M_h be the supremum of f on $[b, b+h]$. Notice that I have to write "supremum" rather than "maximum". (Why?) Using that f is continuous at b (and only at b), prove that $\lim_{h \to 0^+} M_h = f(b)$.

   Note: This is an "ε-δ proof".

(d) Prove Theorem FTC 1' above.

Bonus problems: the definition of log

Up to this point, we have been careless when talking about powers and exponentials (and as a consequence logarithms). We pretend that the quantity a^b makes sense for every $a > 0$ and for every $b \in \mathbb{R}$. However, if we are honest, we only have a good way to define it when $b \in \mathbb{Q}$. How do we actually define 2^π, for example? The solution is to *define*

$$a^b = e^{b \ln a}.$$

Of course, for this to make sense, we have to be able to define first the following two functions

$$E(x) = e^x, \qquad L(x) = \ln x,$$

and moreover, we have to define the functions E and L rigorously *from scratch*, without assuming we already know what exponentials and logarithms are, or that they are well-defined, or their properties.

There are various standard ways to do this. We can define the function E using power series or differential equations, and then define L as its inverse. Or we can define the function L using integrals and define E as its inverse. This collection of problems will guide you through the last option. With the tools you have at the moment, this is surprisingly not too complicated.

Remember: in the next questions you are not allowed to assume that powers or logarithms are even defined yet, or that they have any of the properties we normally assume they have. Everything will have to be proven from the definition.

10. (a) For any integer $n \neq 1$, it is easy to construct an antiderivative of $f(x) = x^n$ as another power times a numerical coefficient. This is well-defined without having to do anything else. But this does not work when $n = -1$. Why?

    (b) Define a new function L with the properties $L(1) = 0$ and $L'(x) = \frac{1}{x}$. Define this function as an integral.

    (c) Verify that this function is well-defined and that it satisfies the two properties above. What is its domain? (Cite any theorem you are using.)

11. Since L is one-to-one, it has an inverse. We define E to be the inverse of L. Obtain a formula for its derivative E'.

12. Prove that for every $x, y > 0$, $L(xy) = L(x) + L(y)$.

    Hint: Fix $y > 0$. Consider the functions $f(x) = L(xy)$ and $g(x) = L(x) + L(y)$. Calculate $f'(x)$ and $g'(x)$. Then...

    Note: All other properties of logarithms can be proven in a similar manner.

13. Use your answer to Question (12) and the fact that E and L are inverses of each other to prove that $E(x + y) = E(x)E(y)$ for all x, y.

    Note: In general, every property of L gives us a property of E. (Or the other way around if we had started with E instead of with L.)

Some answers and hints

1. (a) $\frac{1}{4}x^4 + C$
   (b) $\frac{2}{3}x^{3/2} + C$
   (c) $\frac{-1}{2x^2} + C$
   (d) $\frac{x^4}{4} - \frac{2x^3}{3} + \frac{7x^2}{2} - 5x + C$
   (e) $\frac{(3x+7)^{11}}{33} + C$
   (f) $\sin x + C$
   (g) $-\frac{3}{2}\cos(2x) + C$
   (h) $-\frac{5}{2}e^{-2x} + C$
   (i) $\frac{1}{9(7-6x)^3} + C$
   (j) $\frac{x^3}{3} + x^2 + C$
   (k) $\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C$
   (l) $\ln|x| + C$
   (m) $-\frac{3}{4}(5-2x)^{2/3} + C$
   (n) $\tan x + C$
   (o) $\tan x - x + C$

2. (a) $F'(x) = \frac{1}{x^3+1}$
   (b) $F'(x) = \frac{2x}{x^6+1}$
   (c) $F'(x) = -2x\sin(\cos(x^2))$
   (d) $F'(x) = -(\sin x)\ln(1+\cos^2 x) - (\cos x)\ln(1+\sin^2 x)$

3. (a) $\frac{1}{6}$
   (b) $\frac{e^6 - e^2}{2}$
   (c) 2

4. (a) 32/3
   (b) 37/12
   (c) 18

5. $\frac{3}{4} + \ln 2$

6. $y = (4 + 2\sin 8)x - 2\sin 8$

7. $(a^2 - b^2)(k - c)$

8. F has a local maximum at $x \approx -3.1$, a local minimum at $x = -2$, and a local maximum at $x \approx 0.5$.