MAT 137Y – Practice problems
Unit 6 : Applications of derivatives and limits

Related rates

1. We are inflating a spherical balloon. At what rate is the volume of the balloon changing when the radius is increasing at $3cm/s$ and the volume is $100cm^3$?

2. A cop's radar unit is parked 200m from a long straight stretch of a highway. Down the highway, 200m from the point on the highway closest to the officer, is an emergency phone. The cop points the radar gun at the phone. A car passes the phone and, at the moment, the radar unit indicates that the *distance between the cop and the car* is increasing at 72 km/h. The posted speed limit is 90 km/h. Can the cop fine the driver?

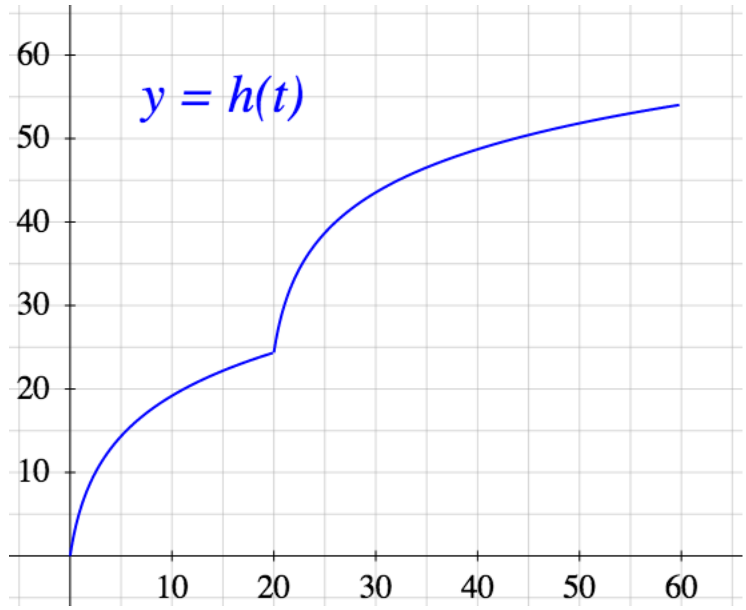
3. A woman walks along a straight path at a speed of 2 m/s. A light is located on the ground 100 meters from the path and is kept focused on her. At what rate is the light rotating when the woman is 75 meters from the point on the path closest to the light?

Applied optimization

4. You have the opportunity to enter a new business manufacturing luxury Anacleto boxes. An Anacleto box is a square open box: the bottom is a square, the four sides are identical rectangles, and there isn't anything on the top. The box should have a volume of 1000 cm^3. The material for the bottom costs \$2 per cm^2 and the material for each of the four sides costs \$3 per cm^2. You know you will be able to sell each box for \$1,250. Is this a profitable enterprise?

5. A farmer wants to hire workers to pick 1600 bags of beans. Each worker can pick 10 bags per hour and is paid \$1.00 per bag. The farmer must also pay a supervisor \$20 per hour while the picking is in progress. She has additional miscellaneous expenses of \$8 per worker (but not for the supersivor). How many workers should she hire to minimize the total cost? What will the cost per bag picked be?

6. You are the CEO of a company that wants to run a commercial during the final of the Stanley Cup. The ad costs \$50,000 per second. Your market research team has produced the graph below. In it, $h(t)$ predicts the extra sales, in hundreds of thousands of dollars, that a commercial of length t seconds will produce. You can buy a commercial of any length between 0 and 60 seconds. Decide how long the commercial should be.



Indeterminate forms and L'Hôpital's Rule

7. Compute the following limits

(a) $\lim_{x\to 1} \frac{\ln x}{x^3-1}$

(b) $\lim_{x\to 0} \frac{x(e^{2x}-1)}{\sin^2 x}$

(c) $\lim_{x\to\infty} \left(x^7+3x^4+2\right)e^{-x}$

(d) $\lim_{x\to\infty} \left[x-\sqrt{x^2+3x}\right]$

(e) $\lim_{x\to-\infty} \left[x-\sqrt{x^2+3x}\right]$

(f) $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$

(g) $\lim_{x\to 0} (\cos x)^{1/x^2}$

(h) $\lim_{x\to\infty} (x+1)^{1/\ln x}$

(i) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt[3]{1+x}}{\sqrt{1+x}-\sqrt[4]{1+x}}$

(j) $\lim_{x\to 0} \left(\frac{1}{x\sin x}-\frac{1}{x\tan x}\right)$

8. Video 6.11 demonstrates that 1^∞ is a limit indeterminate form by constructing examples of functions f and g such that $\lim_{x\to a} f(x)=1$, $\lim_{x\to a} g(x)=\infty$, and $\lim_{x\to a} f(x)^{g(x)}$ is any (positive) number we want. Use the same argument to demonstrate that the following are limit-indeterminate forms:

(a) $\frac{\infty}{\infty}$ (b) $0\cdot\infty$ (c) $\infty-\infty$ (d) 0^0

9. Consider the following FALSE theorem and BAD proof.

False theorem
Let h be a function defined on an open interval I. Assume h is differentiable on I. Then h' is continuous on I.

Bad proof

Let $a \in I$. By definition, $h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$.
Since h is continuous, the limit of the numerator is 0. The limit of the denominator is also 0. Since h is differentiable, I can apply L'Hôpital's Rule.

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \overset{L'H}{=\!=} \lim_{x \to a} \frac{h'(x) - 0}{1 - 0} = \lim_{x \to a} h'(x).$$

I have proven that $h'(a) = \lim_{x \to a} h'(x)$. By definition, h' is continuous. $\square$

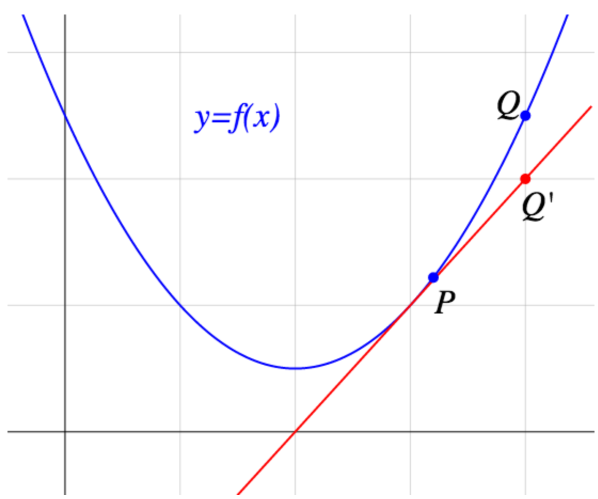
(a) Explain the error in the proof.

(b) "Fix" the theorem. (In other words, modify the statement of the theorem a little bit, either changing the hypotheses or the conclusion, so that it is true. There may be more than one way to do it.)

Concavity

10. (This follows from Video 6.13.) Let f be a differentiable function defined on an open interval I. We will compare the following two concepts:

- Recall that f is concave-up on I when f' is increasing on I.
- We say that f is green when it is differentiable and "the graph of f stays above its tangent lines". This means that for every two different points P and Q on the graph of f, if Q' is a point on the line tangent to the graph of f at P, and Q' has the same x-coordinate as Q, then the y-coordinate of Q is larger than the y-coordinate of Q'.



(a) Show that "f is green" is equivalent to the following condition:

$$\forall a, b \in I, \quad a < b \implies f'(a) < \frac{f(b) - f(a)}{b - a} < f'(b)$$

Hint: One inequality comes from assuming P is to the left of Q, and the other one from assuming P is to the right of Q.

(b) Use the MVT to prove that if f is concave-up, then it is green.

(c) Prove that if f is green, then it is concave-up.

11. Construct a polynomial h that satisfies all of the following conditions at once:

- $h''(0) = 0$
- h is concave up on $(-\infty, 1]$
- h is concave down on $[1, \infty)$

Asymptotes

12. Find all the asymptotes of the following functions:

(a) $f(x) = \dfrac{x^4 - 1}{x(x-1)(x+1)^2}$.

(b) $f(x) = \ln(4x - x^2)$

13. An elementary function is any function you can construct with composition, addition, subtraction, product, and quotient of polynomials, roots, exponentials, logarithms, trigonometric functions, and inverse trigonometric functions. An elementary function cannot be defined piecewise.

Construct an elementary function that has both a horizontal asymptote and a slant asymptote.

Bonus problems: a proof of L'Hôpital's Rule

L'Hôpital's Rule is actually a collection of theorems, depending on the kind of limit (at a real number, a side limit, at ∞, or at $-\infty$) and on the kind of indeterminate form ($0/0$ or ∞/∞). You are going to prove one specific case:

> **Theorem 1.** Let $a \in \mathbb{R}$. Let f and g be functions defined at least on an interval $(a, a+p)$ for some $p > 0$.
> IF
>
> 1. f and g are differentiable on $(a, a+p)$
> 2. g and g' are never 0 on $(a, a+p)$
> 3. $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$
> 4. $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists, or is ∞, or $-\infty$
>
> THEN $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$.

First, you will need to prove a Lemma.

14. Let $a < b$. Let H be a function defined on (a, b). For each $x \in (a, b)$ let c_x be a real number such that $a < c_x < x$. Assume that $\lim_{x \to a^+} H(x)$ exists, or is ∞, or $-\infty$. Prove that $\lim_{x \to a^+} H(c_x) = \lim_{x \to a^+} H(x)$.

15. Under the conditions of Question 14, show with an example that $\lim_{x \to a^+} H(c_x)$ may exist even if $\lim_{x \to a^+} H(x)$ does not exists, is not ∞, and is not $-\infty$.

16. Let's go back to Theorem 1. Assume that f and g are also defined at a and that $f(a) = g(a) = 0$. With this extra hypothesis, prove Theorem 1.

    Hint: For each $x \in (a, a+p)$, apply Cauchy's MVT (Question 6 on the Practice Problems for Unit 5) to f and g on $[a, x]$. Then use Question 14 on this page for $H(x) = \frac{f'(x)}{g'(x)}$.

17. Now prove Theorem 1 without any extra hypotheses.

    Hint: Define a new function $\widehat{f}$ as $\widehat{f}(x) = f(x)$ for all $x \in (a, a+p)$ and $\widehat{f}(a) = 0$. Do the same thing for g. Then use Question 16 on $\widehat{f}$ and $\widehat{g}$.

18. Review the four hypotheses of Theorem 1 and identify exactly where in your proof each one of them was needed. If you did not use one of the hypotheses, then your proof cannot possibly be correct.

Some answers and hints

1. At a rate of about $312cm^3/s$.
2. Yes. The driver's speed is $\sqrt{2} \cdot 72km/h$, which is higher than $90km/h$.
3. At a rate of $0.0128rad/s$.
4. The materials for the box will cost a minimum of $600\sqrt[3]{9} \approx 1,248$ dollars. This leaves a very small profit margin. Since there will be other costs (cost of labour, distribution, publicity, ...) this is unlikely to be profitable.
5. The farmer should hire 20 workers. The total cost per bag picked will be \$1.20.
6. A commercial that runs for about 35 seconds.
7. (a) $1/3$
   (b) 2
   (c) 0
   (d) $-3/2$
   (e) $-\infty$
   (f) $-2/\pi$
   (g) $e^{-1/2}$
   (h) e
   (i) $2/3$
   (j) $1/2$

9. (b) We can only conclude that
   - either h' is continuous at a,
   - or h' has an essential discontinuity at a. This means that $\lim_{x \to a} h'(x)$ does not exist, is not ∞, and is not $-\infty$.

11. It is easier to figure out first what h'' should be.

    We need $h''(0) = 0$ but h'' stays positive to the left and right of 0, and $h''(1) = 0$ with h'' changing sign at 1. The easiest choice is $h''(x) = x^2(1 - x)$.

12. (a) $y = 1$, $x = -1$, $x = 0$.
    (b) $x = 0$, $x = 4$.

13. You can verify your answer by sketching with desmos.