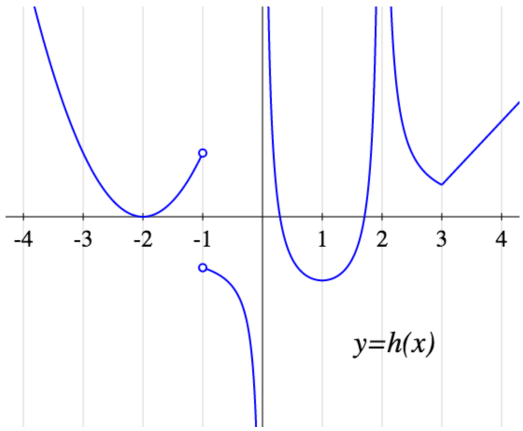
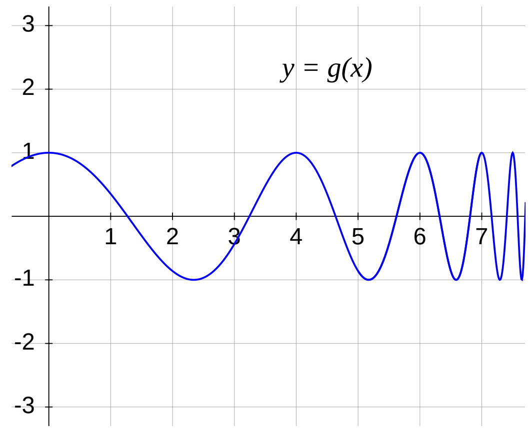
MAT 137Y – Practice problems
Unit 3 : Derivatives

1. Write the equation of the line tangent of the graph of the function f, given by $f(x) = x^3 + 3$, at the point with x-coordinate 2.

2. Below are the graphs of the functions g and h. Sketch the graphs of their derivatives g' and h'.





3. The functions G and H have both domain $\mathbb{R}$. They are continuous everywhere. They satisfy $G(0) = 0$ and $H(0) = 0$. Moreover, $G' = g$ and $H' = h$, where g and h are the functions in Question 2. Sketch the graphs of G and H.

4. Compute the derivative of the following functions:

   (a) $f(x) = 7x^2 - 2x + 1$

   (b) $f(x) = \dfrac{x^2+2}{x^2-2}$

   (c) $f(x) = x^3 \tan(2x+1)$

   (d) $f(x) = \sqrt{\dfrac{x+1}{x-1}}$

   (e) $f(x) = \sin^2 x + \sin x^2 + \sin(2x) + \sin^2 x^2$

   (f) $f(x) = \dfrac{1 + x\sin x}{x + \cos x}$

   (g) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x+1}}}$

5. Let $a \in \mathbb{R}$. Let f be a function which is differentiable at a. Assume that f is always positive. We define a new function g by $g(x) = \dfrac{1}{f(x)}$. Prove that g is differentiable at a and that
$$g'(a) = \frac{-f'(a)}{(f(a))^2}.$$
Write a proof directly from the definition of derivative as a limit without using any of the differentiation rules (such as quotient rule or chain rule).

6. For this question, don't use a calculator. You are going to learn to use linear approximations to estimate values of functions. We have not taught you this in the videos, but that is the point: if you understand what a derivative and a tangent line are, and if you draw a sketch for each part of the question, then everything will be very simple.

(a) Consider the function $h(x) = \sqrt[3]{x}$. Write the equation of the line tangent to $y = h(x)$ at the point with x-coordinate 1. We will call this line L.

(b) Consider the point on L with x-coordinate 1.1 and the point on the graph of h with x-coordinate 1.1. Are their y-coordinates close to each other? Use this to obtain an approximate value for $\sqrt[3]{1.1}$.

(c) Let's say we use the same method (with the same line L) to approximate $\sqrt[3]{1.05}$ and $\sqrt[3]{1.2}$. In which case would we get a smaller error?

(d) Without using the above method, what do you think the approximate value of $\sqrt[3]{28}$ is? Now, using the same line L as above, what value do you get? You will see that this is a very bad approximation. Why didn't it work?

(e) Now you will try the whole thing by yourself. Use a similar method to obtain an approximate value for $\sqrt{3.9}$.

7. Let $c \in \mathbb{R}$. Let g be a function with domain $\mathbb{R}$. We know the following:

- $g(0) = 0$
- g is differentiable at 0 and $g'(0) = c$
- for every $x, y \in \mathbb{R}$, $g(x + y) = g(x) + 2xy + g(y)$.

Prove that g is differentiable everywhere and find a formula for g'.

8. We know the following about the function f:

$$f(0) = 0, \quad f'(0) = 2, \quad f''(0) = 1.$$

We define a new sequence of functions:

$$\begin{aligned} g_2 &= f \circ f \\ g_3 &= f \circ f \circ f \\ g_4 &= f \circ f \circ f \circ f \\ g_5 &= f \circ f \circ f \circ f \circ f \\ &\dots \end{aligned}$$

Compute the following:

(a) $g_3(0)$

(b) $g_{100}(0)$

(c) $g_3'(0)$

(d) $g_{100}'(0)$

(e) $g_3''(0)$

(f) $g_{100}''(0)$

Note: There are various ways to solve this problem. Depending how you do it, you may find the following identity useful. For any $a \neq 1$ and any positive integers $n < m$:

$$a^n + a^{n+1} + a^{n+2} + a^{n+3} + a^{n+4} + \ldots + a^m = \frac{a^{m+1} - a^n}{a - 1}.$$

9. Consider the function $y = g(x)$ defined implicitly by the equation $x^3 + xy^2 = y^3 + 1$ near $x = 0$. Calculate $g(0)$, $g'(0)$, and $g''(0)$.

10. The most common way to derive formulas for the derivatives of the six trig functions is the one you learned in the videos: we obtain the derivative of sin or cos from the definition ("the long way") and then we use the quotient rule to derive the rest. But we could have done it in other ways.

    For the purpose of this problem, assume you know the basic differentiation rules (linearity, power, product, quotient, and chain) but that you do not know yet any of the formulas for derivatives of trig functions.

    (a) Obtain a formula for the derivative of tan directly from the definition of derivative as a limit.

    Hint: Write $\tan x = \dfrac{\sin x}{\cos x}$ and use the formulas for the sine of the sum and the cosine of the sum. This is similar to the derivation in Video 3.12.

    (b) Use your answer to Question 10a and implicit differentiation on

    $$\sec^2 x = 1 + \tan^2 x$$

    to obtain a formula for the derivative of sec.

    (c) Use your answer to Question 10b to obtain a formula for the derivative of cos.

    (d) Use your answer to Question 10c and the equations

    $$\cos x = \sin\left(\frac{\pi}{2} - x\right), \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

    to obtain a formula for the derivative of sin.

    Note: Strictly speaking, this derivation only works in the domain of tan, but sin and cos are defined (and differentiable) on all of $\mathbb{R}$. For the purpose of this question we will ignore this point.

11. Consider the function $h(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$

    (a) Compute $h'(x)$ for $x \neq 0$.

    (b) Using the definition of derivative as a limit, compute $h'(0)$.

    (c) Calculate $\lim\limits_{x \to 0} h'(x)$.

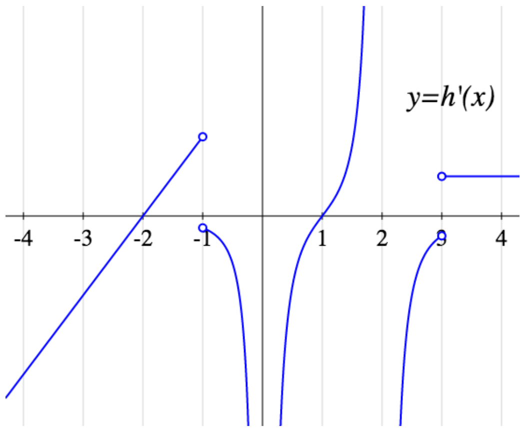
    (d) Show that h is differentiable everywhere, but h' is not continuous at 0.

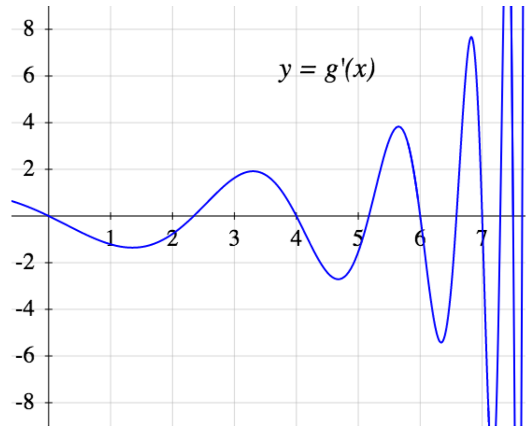
    Note: You have already done all the work you need; this last question is just a matter of realizing what it means.

Some answers and hints

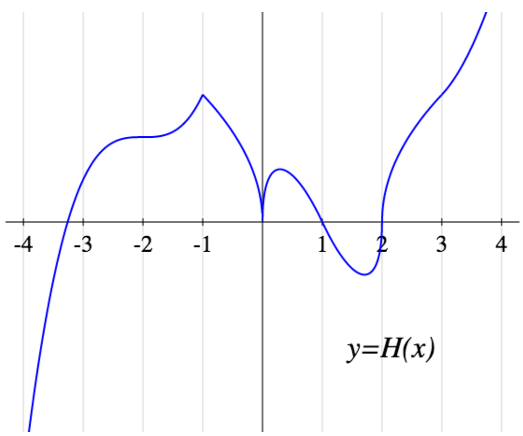
1. $y = 12x - 13$

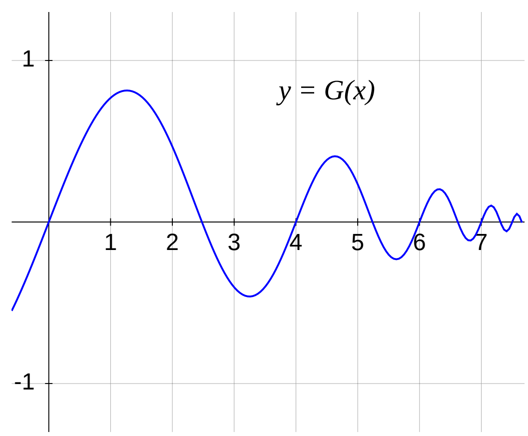
2.





3.





Notice that the graph of H becomes virtually "vertical" at $x = 0$ and at $x = 2$.

4. (a) $f'(x) = 14x - 2$

   (b) $f'(x) = \dfrac{-8x}{(x^2 - 2)^2}$

   (c) $f'(x) = 3x^2 \tan(2x + 1) + 2x^3 \sec^2(2x + 1)$

   (d) $f'(x) = \dfrac{-1}{|x - 1|\sqrt{x^2 - 1}}$

   (e) $f'(x) = \sin(2x) + 2x \cos x^2 + 2\cos(2x) + 2x \sin(2x^2)$

   (f) $f'(x) = \dfrac{x^2 \cos x + x + \sin x \cos x + \sin x - 1}{(x + \cos x)^2}$

(g) $f'(x) = \dfrac{1 \; + \; 2\sqrt{x+1} \; + \; 4\sqrt{x+1}\,\sqrt{x+\sqrt{x+1}}}{8\,\sqrt{x+1}\,\sqrt{x+\sqrt{x+1}}\,\sqrt{x+\sqrt{x+\sqrt{x+1}}}}$

5.
$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{\dfrac{1}{f(x)} - \dfrac{1}{f(a)}}{x-a} = \dots$$

Then use that $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ and that f is continuous at a.

There are three common errors students make in this proof. First, if your proof is correct, you need to use at some point that the function f was continuous. Make sure to indicate when you are using this, and why you know it is continuous. Second, if you start your proof computing a formula for $g'(x)$ and only at the end you substitute $x = a$, your proof is entirely wrong. (Why?) Third, your proof must have noted at some point that f is never 0.

6. (a) $y = 1 + \dfrac{1}{3}(x-1)$

   (b) $\sqrt[3]{1.1} \approx 1 + \dfrac{0.1}{3}$

   (c) The error will be smaller when we estimate $\sqrt[3]{1.05}$.

   (d) Draw a picture: the points with $x = 28$ on the graph of h and on the line L are very far from each other, because 28 is very far from 1.

   (e) $\sqrt{3.9} \approx 1.95$

7.
$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \ldots = \ldots = 2x + c$$

8. $g_{100}(0) = 0, \quad g'_{100}(0) = 2^{100}, \quad g''_{100}(0) = 2^{199} - 2^{99}.$

9. $g(0) = -1, \quad g'(0) = \dfrac{1}{3}, \quad g''(0) = \dfrac{-2}{9}.$

10. You know what your final answers should be. Just make sure that at each point you are only using things that have already been proven.

11. (a) For $x \neq 0$, $\quad h'(x) = 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}$

    (b) $h'(0) = 0$

    (c) $\lim_{x\to 0} h'(x)$ DNE.