MAT 137Y – Practice problems
Unit 2 : Limits and continuity

1. Below is the graph of the function $f$:

Compute the following limits

(a) $\lim_{x \to 2} f(x)$  
(b) $\lim_{x \to 0} f(f(x))$  
(c) $\lim_{x \to -3} f(f(x))$  
(d) $\lim_{x \to 0} f(2 \sec x)$  
(e) $\lim_{x \to 2} (f(x))^2$

2. Given a real number $x$, we defined the \textit{floor of} $x$, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to $x$. For example, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, and $\lfloor -0.5 \rfloor = -1$.

(a) Sketch the graph of this function. At which points is the function $f(x) = \lfloor x \rfloor$ continuous? Which discontinuities are removable and which ones are non-removable?

(b) Consider the function $h(x) = \lfloor \sin x \rfloor$. Show that $h$ has exactly one removable and one non-removable discontinuity inside the interval $(0, 2\pi)$.

3. Below is the graph of the function $g$:
For clarification, when $-1 < x < 0$, $g(x)$ “oscillates” between 0 and 1; as $x$ approaches 0 from the left, these oscillations become faster and faster. The behaviour is similar to that of the function $f(x) = \sin(\pi/2x)$, which you can see on Video 2.2.

Find the following limits:

(a) \( \lim_{x \to 0^+} g(x) \)
(b) \( \lim_{x \to 0^+} \lfloor g(x) \rfloor \)
(c) \( \lim_{x \to 0^+} g(\lfloor x \rfloor) \)
(d) \( \lim_{x \to 0^-} g(x) \)
(e) \( \lim_{x \to 0^-} \lfloor g(x) \rfloor \)
(f) \( \lim_{x \to 0^-} \lfloor g(\lfloor x \rfloor) \rfloor \)

4. Compute the following limits

(a) \( \lim_{x \to 1} \frac{x + 1}{x + 2} \)
(b) \( \lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4} \)
(c) \( \lim_{x \to 1} \frac{\sqrt{x + 3} - 2}{x - 1} \)
(d) \( \lim_{x \to 0} \frac{\sin(3x)}{\sin(2x)} \)
(e) \( \lim_{x \to \infty} \frac{x^3 + 2x^2 + 1}{5x^3 + 6x - 1} \)
(f) \( \lim_{x \to -\infty} \frac{\sin^{10}(2\sin^{10}(3x))}{x^{100}} \)

5. Write the formal definition of the following concepts:

(a) \( \lim_{x \to a} f(x) = L \)
(b) \( \lim_{x \to a} f(x) \) exists
(c) \( \lim_{x \to a} f(x) \) does not exist
(d) \( \lim_{x \to a} f(x) = -\infty \)
(e) \( \lim_{x \to a}^+ f(x) = L \)
(f) \( \lim_{x \to a} f(x) = \infty \)
(g) \( \lim_{x \to a^-} f(x) = \infty \)
(h) \( \lim_{x \to a^-} f(x) = -\infty \)

6. Prove the following claims directly from the formal definitions.

(a) \( \lim_{x \to 2} (4x + 1) = 9 \)
(b) \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \)
(c) \( \lim_{x \to 1} x^3 = 1 \)
(d) \( \lim_{x \to 1} \frac{1}{x^2 + 1} = \frac{1}{2} \)
(e) \( \lim_{x \to 0} \frac{x}{\lfloor x \rfloor} \) does not exist
(f) \( \lim_{x \to 1^+} \frac{1}{1 - x} = -\infty \)

7. Let \( a, L, M \in \mathbb{R} \). Let \( f \) be a function defined, at least, on an interval centered at \( a \), except maybe at \( a \). Prove that

\[
\text{IF } \lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M \quad \text{THEN } \lim_{x \to a} [f(x) - g(x)] = L - M.
\]

Write a proof directly from the formal definitions, without using any of the limit laws.

8. Let \( a \in \mathbb{R} \). Let \( f \) be a function defined at least on an interval centered at \( a \), except possibly at \( a \). Prove that

\[
\text{IF } \lim_{x \to a} f(x) = \infty \quad \text{THEN } \lim_{x \to a} \frac{1}{f(x)} = 0.
\]
Write a proof directly from the formal definitions, without using any of the limit laws.

9. Construct a function \( f \) with domain \( \mathbb{R} \) such that \( \lim_{x \to 0} f(x) = 0 \) but \( \lim_{x \to 0} f(f(x)) \neq 0 \).

10. Prove Theorem 3 on Video 2.16. More specifically:

Let \( a, L \in \mathbb{R} \). Let \( f \) be a function defined, at least, on an interval centered at \( a \), except maybe at \( a \). Let \( g \) be a function defined at least on an interval centered at \( L \). Prove that

\[
\text{IF } \lim_{x \to a} f(x) = L \text{ and } g \text{ is continuous at } L \quad \text{THEN } \lim_{x \to a} g(f(x)) = g(L).
\]

Write a proof directly from the formal definitions, without using any of the limit laws.

11. Use the Intermediate Value Theorem to prove that the equation

\[
\sin x = 2 \cos^2 x + 0.5
\]

has at least one solution.

12. Use the Squeeze Theorem to explain why \( \lim_{x \to 0} x \cos \frac{1}{x} \) exists, even though \( \lim_{x \to 0} \cos \frac{1}{x} \) does not exist. Explain why the same argument does not work for \( \lim_{x \to 0} xe^{1/x^2} \).
Bonus question:
Do you really understand the definition of limit?

13. Let \( f \) be a function. Let \( a, L \in \mathbb{R} \). Assume that \( f \) is defined on some open interval around \( a \), except maybe at \( a \). Below is a list of nine statements.

a. \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

b. \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

c. \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies 0 < |f(x) - L| < \varepsilon \).

d. \( \forall \varepsilon \geq 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

e. \( \forall \varepsilon > 0, \exists \delta \geq 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

f. \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| \leq \varepsilon \).

g. \( \forall \delta > 0, \exists \varepsilon > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

h. \( \forall \delta > 0, \exists \varepsilon > 0 \text{ such that } 0 < |x - a| < \varepsilon \implies |f(x) - L| < \delta \).

i. \( \exists \delta > 0 \text{ such that } \forall \varepsilon > 0, 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

Match each of the statements above to one of the following (there may be repeats):

A. Every function satisfies this statement.

B. There isn’t any function which satisfies this statement.

C. This statement is (equivalent to) the definition of \( \lim_{x \to a} f(x) = L \).

D. This statement is (equivalent to) the definition of “\( f \) is continuous at \( a \)”.

E. This statement means that \( \lim_{x \to a} f(x) = L \) and that, in addition, \( f \) does not take the value \( L \) anywhere on some interval centered at \( a \), except maybe at \( a \).

F. This statement is equivalent to saying that \( f \) must be constantly equal to \( L \) on an interval centered at \( a \), except maybe at \( a \).

G. This statement means that \( f \) is bounded on every interval centered at \( a \).
Some answers and hints

1. (a) DNE  (b) -2  (c) -1  (d) 2  (e) 4

2. (a) \( f \) is discontinuous at \( a \) when \( a \in \mathbb{Z} \). \( f \) is continuous everywhere else. All the discontinuities are non-removable.

(b) \( g \) has a removable discontinuity at \( \frac{\pi}{2} \) and a non-removable discontinuity at \( \pi \).

3. (a) 2  (b) 1  (c) 1.5  (d) DNE  (e) DNE  (f) 0  (g) 0.5

4. (a) \( \frac{2}{3} \)  (d) \( \frac{3}{2} \)  (g) 4

(b) \( \frac{7}{4} \)  (e) \( \frac{1}{5} \)  (h) DNE

(c) \( \frac{1}{4} \)  (f) \( \infty \)  (i) \( 2^{100} \cdot 3^{100} \)

5. There are various equivalent ways to write each definition. The parts in blue (and only the parts in blue) are often omitted and are considered implicit.

(a) \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } (\forall x \in \mathbb{R},) \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \)

(b) \( \exists L \in \mathbb{R} \text{ such that } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } (\forall x \in \mathbb{R},) \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \)

(c) \( \exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in \mathbb{R} \text{ such that } \quad |0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon \)

(d) \( \forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in \mathbb{R} \text{ such that } \quad |0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon \)

(e) \( \forall \varepsilon > 0, \exists \varepsilon > 0 \text{ such that } (\forall x \in \mathbb{R},) \quad a < x < a + \delta \implies |f(x) - L| < \varepsilon \)

(f) \( \forall M \in \mathbb{R}, \exists \delta > 0 \text{ such that } (\forall x \in \mathbb{R},) \quad 0 < |x - a| < \delta \implies f(x) > M \)

(g) \( \forall M \in \mathbb{R}, \exists \delta > 0 \text{ such that } (\forall x \in \mathbb{R},) \quad a - \delta < x < a \implies f(x) < M \)

(h) \( \forall \varepsilon > 0, \exists K \in \mathbb{R} \text{ such that } (\forall x \in \mathbb{R},) \quad x > K \implies |f(x) - L| < \varepsilon \)

(i) \( \forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ such that } (\forall x \in \mathbb{R},) \quad x < K \implies f(x) > M \)

6. (a) This is similar to the proof in Video 2.7.

(b) WTS: \( \forall \varepsilon > 0, \exists K \in \mathbb{R} \text{ such that } \forall x \in \mathbb{R}, \quad x > K \implies \left| \frac{1}{x^2} - 0 \right| < \varepsilon \)

- Fix \( \varepsilon > 0 \)

- Take \( K = \frac{1}{\sqrt{\varepsilon}} \).

- Fix \( x \in \mathbb{R} \). Assume \( x > K \). I need to verify that \( \frac{1}{x^2} < \varepsilon \).

\[
\frac{1}{x^2} < \frac{1}{K^2} = \varepsilon.
\]

(c) This is similar to the proof in Video 2.8

(d) WTS: \( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \quad 0 < |x - 1| < \delta \implies \left| \frac{1}{x^2 + 1} - \frac{1}{2} \right| < \varepsilon \)

- Fix \( \varepsilon > 0 \)
• Take $\delta = \min\{1, 2\varepsilon/3\}$. Thus $\delta \leq 1$ and $\delta \leq 2\varepsilon/3$.

• Fix $x \in \mathbb{R}$. Assume $0 < |x - 1| < \delta$. I need to verify that $\left| \frac{1}{x^2 + 1} - \frac{1}{2} \right| < \varepsilon$.

By assumption, $0 \leq 1 - \delta < x < 1 + \delta \leq 2$. Thus $|1 + x| < 3$.

In addition $\frac{1}{x^2 + 1} \leq 1$.

\[
\left| \frac{1}{x^2 + 1} - \frac{1}{2} \right| = \frac{|x + 1||x - 1|}{2(x^2 + 1)} < \frac{3\delta}{2 \cdot 1} \leq \varepsilon.
\]

(e) This is somewhat similar to the proof in Video 2.9.

(f) WTS $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \ 1 < x < 1 + \delta \implies \frac{1}{1 - x} < M$

• Fix $M \in \mathbb{R}$

• Next we need to choose $\delta$. It is probably easiest to break this into two cases.
  - If $M > 0$, take $\delta = 1$ for example.
  - If $M \leq 0$ take $\delta = \frac{1}{|M|}$

• Fix $x \in \mathbb{R}$. Assume $1 < x < 1 + \delta$. I need to verify that $\frac{1}{1 - x} < M$.

... 

(Pay careful attention to the signs. Sometimes you will be working with negative numbers.)

7. This proof is very similar to the one in Video 2.11.

8. WTS $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \ 0 < |x - a| < \delta \implies \left| \frac{1}{f(x)} \right| < \varepsilon$

• Fix an arbitrary $\varepsilon > 0$.

• Using $\frac{1}{\varepsilon}$ as the bound in the definition of $\lim_{x \to a} f(x) = \infty$, we can conclude that

  $\exists \delta > 0$ such that $\forall x \in \mathbb{R}, \ 0 < |x - a| < \delta \implies f(x) > \frac{1}{\varepsilon}$

  This is the value of $\delta$ I take.

• Let $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$. I need to verify that $\left| \frac{1}{f(x)} \right| < \varepsilon$.

  This follows immediately from knowing that $f(x) > \frac{1}{\varepsilon} > 0$.

9. This is definitely possible. You will need a function that is not continuous at 0, although being discontinuous at 0 is not enough.

10. I want to prove that

  $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in \mathbb{R}, \ 0 < |x - a| < \delta \implies |g(f(x)) - g(L)| < \varepsilon$. 

• Fix an arbitrary $\varepsilon > 0$.
• First I use this value of $\varepsilon$ in the definition of “$g$ is continuous at $L$” to conclude that

$$\exists \delta_0 > 0 \text{ such that } \forall y \in \mathbb{R}, \ |y - L| < \delta_0 \implies |g(y) - g(L)| < \varepsilon.$$  

Second I use this value of $\delta_0$ “as the epsilon” in the definition of “$\lim_{x \to a} f(x) = L$” to conclude that

$$\exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \ 0 < |x - a| < \delta \implies |f(x) - L| < \delta_0.$$  

This is the value of $\delta$ I take.
• Fix $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$. I need to verify that $|g(f(x)) - g(L)| < \varepsilon$.
  - Since $0 < |x - a| < \delta$, we conclude that $|f(x) - L| < \delta_0$.
  - Since $|f(x) - L| < \delta_0$, we conclude that $|g(f(x)) - g(L)| < \varepsilon$.

11. Consider the function $f$ defined by $f(x) = \sin x - 2 \cos^2 x$. $f$ has domain $\mathbb{R}$ and is continuous everywhere.

$$f(0) = -2 < 0.5, \quad f(\pi/2) = 1 > 0.5.$$  

Therefore, by the Intermediate Value Theorem, $\exists x \in (0, \pi/2)$ such that $f(x) = 0.5$.

12. This is similar to the argument in Video 2.12.

13. A. e  
   B. d  
   C. a, f, h  
   D. b  
   E. c  
   F. i  
   G. g