MAT 137Y – Practice problems
Unit 1 : Logic, notation, definition, and proofs

1. Negate each of the following statements without using any negative words ('no', 'not', 'none', etc):
   (a) "*Every page in this book contains at least one word whose first and last letters both come alphabetically before M.*"
   (b) "*I have a friend all of whose former boyfriends had at least two siblings with exactly three different vowels in their name.*"
   (c) "*If a student in this class likes the musical Cats then they are not my friend.*"

2. What are each of the following sets?
   (a) $\mathbb{N} \cap \mathbb{Z} \cap \mathbb{Q} \cap \mathbb{R}$
   (b) $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q}$
   (c) $(-1, e) \cup [0, \pi]$
   (d) $(-1, e) \cap [0, \pi]$
   (e) $(-e, \pi] \cap \mathbb{Z}$
   (f) $(-1, 2] \cap \varnothing$

3. Consider the following definitions about real numbers x:
   - x is *courageous* when $\forall a > 0, x < a$
   - x is *hard-working* when $\forall a \geq 0, x < a$
   - x is *intelligent* when $\forall a > 0, x \leq a$
   - x is *ambitious* when $\forall a \geq 0, x \leq a$

   Even though all these definitions look different, some of them mean exactly the same. Which of these concepts are equivalent, and which ones are not?

4. Write a definition of the following sets that uses only mathematical notation:
   (a) The set of positive, rational numbers and negative, irrational numbers.
   (b) The set of rational numbers whose numerator and denominator are both odd.
   (c) The set of natural numbers that cannot be written as the sum of two squares of natural numbers.

5. Every professor in the math department is a truthteller (always tells the truth) or a liar (always says false statements). Five professors make the following statements:
   - Alice: "If I am a liar, then so is Bob".
   - Bob: "If I am a liar, then so is Carol."
   - Carol: "If I am a liar, then so is Dave."
   - Dave: "If I am a liar, then so is Eve."
   - Eve: "If I am a liar, then so is Alice."

What is the largest number of them that could possibly be liars?

6. Given two sets A and B of real numbers, we say that B *dominates* A when the following statement is true:

   "For every $a \in A$, there exists $b \in B$ such that $a < b$."

   Find two non-empty sets A and B such that the following three properties are all simultaneously true:

   (i) $A \cap B$ is empty, (ii) A dominates B (iii) B dominates A

7. Consider the following set

$$A = \left\{ x \in \mathbb{R} \ : \ \exists y \in \mathbb{N} \text{ s.t. } x = \frac{y}{y+1} \right\}$$

   Below are various other attempts to define it in different ways. Five of them are fine. The other five are either bad notation or mean something different. Which ones are which?

   (a) $A = \left\{ x \in \mathbb{Q} \ : \ \exists y \in \mathbb{N} \text{ s.t. } x = \frac{y}{y+1} \right\}$

   (b) $A = \left\{ x \in \mathbb{R} \ : \ \exists y \in \mathbb{N} : x = \frac{y}{y+1} \right\}$

   (c) $A = \left\{ x \in \mathbb{R}, y \in \mathbb{N} \ : \ x = \frac{y}{y+1} \right\}$

   (d) $A = \left\{ x = \frac{y}{y+1} \ : \ x \in \mathbb{R}, y \in \mathbb{N} \right\}$

   (e) $A = \left\{ \frac{y}{y+1} \ : \ y \in \mathbb{N} \right\}$

   (f) $A = \left\{ \frac{y}{y+1} \mid y \in \mathbb{N} \right\}$

   (g) $A = \left\{ \frac{y}{y+1} \ , \ y \in \mathbb{N} \right\}$

   (h) $A = \left\{ x \in \mathbb{R} \ : \ x = \frac{y}{y+1} \text{ for some } y \in \mathbb{Z}, y \geq 0 \right\}$

   (i) $A = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1} \right\}$

   (j) $A = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \right\}$

8. This problem is about the definition of periodic function. We assume you already know intuitively what periodic means, and now we want a formal definition. For simplicity, we will restrict ourselves to functions with domain $\mathbb{R}$. A naive (but incorrect) definition of periodic function with period T is

$$f(x+T) = f(x)$$

Without accompanying words, this is not a good definition because it does not introduce the variables x and T and it does not explain their role. For which values of x and T does the above definition have to be valid?

Here is an attempt at a definition, with various ways to complete it:

> **Definition.** Let f be a function with domain $(-\infty, \infty)$. We say that f is periodic when...
>
> (a) For every $x \in (-\infty, \infty)$ and for every $T > 0$, $f(x+T) = f(x)$.
> (b) For every $x \in (-\infty, \infty)$ there exists $T > 0$ such that $f(x+T) = f(x)$.
> (c) There exists $T > 0$ such that $x \in (-\infty, \infty) \implies f(x+T) = f(x)$.
> (d) There exists $T > 0$ such that for every $x \in (-\infty, \infty)$, $f(x+T) = f(x)$.
> (e) For every $T > 0$ there exists $x \in (-\infty, \infty)$ such that $f(x+T) = f(x)$.

One or more of the above are valid ways to complete the definition of periodic function. Identify which ones are correct and which ones are wrong. For any property which is wrong, show it by giving an example of a function which satisfies the property but is not periodic, or an example of a function which is periodic but does not satisfy the property. It is okay to give your examples as equations or as graphs.

9. Write a formal proof for the following statement:

$$\forall a > 0,\ \exists b \in \mathbb{R} \text{ such that } (b + \sin b)a > 7.$$

10. Prove that for every positive integer n

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Answers

1. Notice that there are other correct answers.
   (a) There is a page in this book that contains only words whose first or last letter come alphabetically after L.
   (b) Each of my friends has a former boyfriend who had at most one siblings with exactly three different vowels in their name.
   (c) There is a student in this class who likes the musical Cats and who is my friend.
2. (a) $\mathbb{N}$
   (b) $\mathbb{Q}$
   (c) $(-1, \pi]$
   (d) $[0, e)$
   (e) $\{-2, -1, 0, 1, 2, 3\}$
   (f) $\varnothing$
3. "Courageous", "intelligent", and "ambitious" mean the same. "Hard-working" does not. Hufflepuffs are special.
4. There are many correct answers, but beware of bad notation: see Question 7. Sample correct answers (but there are many others):
   (a) $\{x \in \mathbb{R} \,:\, x > 0, x \in \mathbb{Q}\} \cup \{x \in \mathbb{R} \,:\, x < 0, x \notin \mathbb{Q}\}$
   (b) $\left\{ \dfrac{2n+1}{2m+1} \,:\, n, m \in \mathbb{Z} \right\}$
   (c) $\left\{ n \in \mathbb{N} \,:\, \forall a, b \in \mathbb{N}, n \neq a^2 + b^2 \right\}$
5. Two.
6. This is definitely possible. Once you understand the statements, if you come up with a correct example you will know it is right.
7. (a), (e), (f), (h), (j) are correct and equivalent.
8. (c) and (d) are correct.
9. The structure of the proof should be
   - first fix an arbitrary $a > 0$,
   - second say what b is in terms of a, and
   - third prove the inequality is true.

   It is unlikely your proof is correct without this. Here is a sample correct proof (but not the only one):

Proof:

- Let us fix an arbitrary $a > 0$.
- We take $b = \dfrac{8}{a} + 1$.
- We need to show that $(b + \sin b)a > 7$. Indeed:

$$(b + \sin b)a \;\geq\; (b-1)a \;=\; \frac{8}{a}a = 8 > 7,$$

as needed. $\square$

10. This is a standard proof by induction. Imitate the second proof in Video 1.15.