MAT 137Y: Calculus with proofs
Test 5 - Part B - Sample Solutions

We define a new function h via the equation

$$h(x) = \sum_{n=0}^{\infty} \frac{n!}{(2n)!} x^n$$

Notice that h is defined as a power series. Its interval of convergence is $\mathbb{R}$. (You do not need to prove this.)

1. Find the equation of the line tangent to the graph of h at the point with x-coordinate 0.

Solution:

METHOD 1: h is defined as a power series centered at 0 (with positive radius of convergence), so it is analytic at 0 and its Maclaurin series is itself. Therefore, we can obtain any Maclaurin polynomial of h simply by truncating the power series. In particular, the 1st Maclaurin polynomial is the series truncated at degree 1:

$$P_1(x) \;=\; \sum_{n=0}^{1} \frac{n!}{(2n)!} x^n \;=\; 1 + \frac{1}{2}x$$

And we know the 1st Maclaurin polynomial gives us the tangent line:

$$y = 1 + \frac{1}{2}x.$$

METHOD 2: The tangent line at the point with x-coordinate 0 is given by $y = h(0) + h'(0)x$. We calculate these two values:

$$\begin{aligned} h(x) &= 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \dots & \qquad h(0) &= 1 \\ h'(x) &= \frac{1}{2} + \frac{1}{6}x + \dots & \qquad h'(0) &= \frac{1}{2} \end{aligned}$$

Hence, the tangent line has equation $y = 1 + \frac{1}{2}x$.

2. Calculate the value of the sum

$$S = \sum_{n=0}^{\infty} \frac{(n+2)!}{(2n)!}.$$

Write your answer in terms of values of h and its derivatives.

Solution:

METHOD 1: Notice that

$$S = \sum_{n=0}^{\infty} \left[(n+2)(n+1) \cdot \frac{n!}{(2n)!} \right]$$

We know that, for every $x \in \mathbb{R}$:

$$h(x) = \sum_{n=0}^{\infty} \frac{n!}{(2n)!} x^n$$

I can obtain S by taking the second derivative of $x^2 h(x)$, and then evaluating at $x = 1$. More specifically:

$$x^2 h(x) = \sum_{n=0}^{\infty} \frac{n!}{(2n)!} x^{n+2}$$

$$\frac{d}{dx}\left[x^2 h(x)\right] = \sum_{n=0}^{\infty} \frac{n! \cdot (n+2)}{(2n)!} x^{n+1}$$

$$\frac{d^2}{dx^2}\left[x^2 h(x)\right] = \sum_{n=0}^{\infty} \frac{n! \cdot (n+2)(n+1)}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(n+2)!}{(2n)!} x^n$$

Now I use the product rule repeatedly:

$$\frac{d}{dx}\left[x^2 h(x)\right] = 2x h(x) + x^2 h'(x)$$

$$\frac{d^2}{dx^2}\left[x^2 h(x)\right] = 2h(x) + 2x h'(x) + 2x h'(x) + x^2 h''(x) = 2h(x) + 4x h'(x) + x^2 h''(x)$$

And finally I evaluate at $x = 1$:

$$\boxed{S = 2h(1) + 4h'(1) + h''(1)}$$

METHOD 2: There are many other alternative methods. For example, we can notice that

$$h(x) = \sum_{n=0}^{\infty} \frac{n!}{(2n)!} x^n$$

$$h'(x) = \sum_{n=1}^{\infty} \frac{n \cdot n!}{(2n)!} x^{n-1}$$

$$h''(x) = \sum_{n=2}^{\infty} \frac{n(n-1) \cdot n!}{(2n)!} x^{n-2}$$

and that

$$S \;=\; \sum_{n=0}^{\infty}\left[(n+2)(n+1)\cdot\frac{n!}{(2n)!}\right]$$

Then we notice that

$$(n+2)(n+1) \;=\; n^2+3n+2 \;=\; n(n-1)+4n+2$$

And finally we use a bit of algebra to write S as a linear combination of $h(1)$, $h'(1)$, and $h''(1)$. There are other ways.

3. Find the largest $n \in \mathbb{N}$ such that the limit

$$\lim_{x\to 0} \frac{h(-x^2) - \cos(x)}{x^n}$$

exists, then calculate the limit.

Solution:

To compute this limit I will write the first few terms of the numerator.

$$\begin{aligned}
h(x) &= 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{120}x^3 + \dots \\
h(-x^2) &= 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{120}x^6 + \dots \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\
h(-x^2) - \cos(x) &= \left[\frac{1}{12} - \frac{1}{24}\right] x^4 + (\text{h.o.t.}) = \frac{1}{24}x^4 + (\text{h.o.t.})
\end{aligned}$$

Now let's compute the limit.

- If $n < 4$ then

$$\lim_{x\to 0} \frac{h(-x^2) - \cos(x)}{x^n} = \lim_{x\to 0}\left[\frac{1}{24}x^{4-n} + (\text{h.o.t.})\right] = 0$$

  because $4 - n > 0$.

- If $n = 4$ then

$$\lim_{x\to 0} \frac{h(-x^2) - \cos(x)}{x^4} = \lim_{x\to 0}\left[\frac{1}{24} + (\text{h.o.t.})\right] = \frac{1}{24}$$

- If $n > 4$ then

$$\lim_{x\to 0} \frac{h(-x^2) - \cos(x)}{x^n} = \lim_{x\to 0} \frac{\frac{1}{24} + (\text{h.o.t.})}{x^{n-4}}$$

  which does not exists (because $n - 4 > 0$). The limit would actually be $\pm\infty$, possibly different on the left and on the right.

Therefore, the largest value of n for which the limit exists is $\boxed{n = 4}$, and the limit is $\boxed{L = \frac{1}{24}}$.