MAT 137Y: Calculus with proofs
Test 4 - Part B - Sample solutions

QUESTION 1

Q1 - The problem

For which values of $a, b > 0$ are the following integrals convergent?

(a) $\displaystyle\int_1^\infty \frac{1}{x^a(1+x^b)}\,dx$

(b) $\displaystyle\int_0^1 \frac{1}{x^a(1+x^b)}\,dx$

(c) $\displaystyle\int_0^\infty \frac{1}{x^a(1+x^b)}\,dx$

Q1 - Important notes

- The behaviour of powers of x is different as $x \to \infty$ and as $x \to 0^+$:

$$\lim_{x\to\infty} \frac{x + 2x^2}{3x + 5x^5} = \frac{2}{5}, \qquad \text{but} \qquad \lim_{x\to 0^+} \frac{x + 2x^2}{3x + 5x^5} = \frac{1}{3}$$

  In other words, in the sum $x^a + x^{a+b}$ ...

  - ... the "dominant" term is x^{a+b} as $x \to \infty$
  - ... the "dominant" term is x^a as $x \to 0^+$

- It is not enough to prove the integral is convergent for some values of a and b.

  You also need to prove that it isn't convergent for the rest of the values.

Q1 - Solution

(a) $\boxed{\displaystyle\int_1^\infty \frac{1}{x^a(1+x^b)}\,dx \quad \text{is convergent iff } a+b>1.}$

Proof: The integral is improper at ∞. I want to compare it with $\displaystyle\int_1^\infty \frac{1}{x^{a+b}}\,dx$.

$$\lim_{x\to\infty} \frac{\dfrac{1}{x^a(1+x^b)}}{\dfrac{1}{x^{a+b}}} = \lim_{x\to\infty} \frac{x^{a+b}}{x^a + x^{a+b}} = \lim_{x\to\infty} \frac{1}{\dfrac{1}{x^b}+1} = 1$$

By LCT, $\displaystyle\int_1^\infty \frac{1}{x^a(1+x^b)}\,dx$ is convergent iff $\displaystyle\int_1^\infty \frac{1}{x^{a+b}}\,dx$ is convergent.
We know the second integral is convergent iff $a+b>1$.

(b) $\boxed{\displaystyle\int_0^1 \frac{1}{x^a(1+x^b)}\,dx \quad \text{is convergent iff } a<1.}$

Proof: The integral is improper at 0^+. I want to compare it with $\displaystyle\int_0^1 \frac{1}{x^a}\,dx$.

$$\lim_{x\to 0^+} \frac{\dfrac{1}{x^a(1+x^b)}}{\dfrac{1}{x^a}} = \lim_{x\to 0^+} \frac{1}{1+x^b} = 1$$

By LCT, $\displaystyle\int_0^1 \frac{1}{x^a(1+x^b)}\,dx$ is convergent iff $\displaystyle\int_0^1 \frac{1}{x^a}\,dx$ is convergent.
We know the second integral is convergent iff $a<1$.

(c) $\boxed{\displaystyle\int_0^\infty \frac{1}{x^a(1+x^b)}\,dx \quad \text{is convergent iff } (a+b>1 \text{ AND } a<1\ )}$

We can also write the condition as $1-b<a<1$.

Proof:

$$\int_0^\infty \frac{1}{x^a(1+x^b)}\,dx = \int_0^1 \frac{1}{x^a(1+x^b)}\,dx + \int_1^\infty \frac{1}{x^a(1+x^b)}\,dx$$

The first integral is convergent if and only if the other two integrals are convergent independently.
Then I use the result from Questions 1a and 1b.

QUESTION 2

Q2 - The problem

Prove that IF a sequence is divergent to ∞, THEN it is bounded below.
Suggestion: You know that every convergent sequence is bounded. Revisit the proof.

Q2 - Important notes

This proof is extremely similar to the one in Video 11.5. You only need to modify a few things. You won't get any credit for copying down the proof of Video 11.5. You only get credit for understanding what needs to be modified and *writing it correctly.* This means you could write a proof that is partially right and still get 0 points.

In particular, if you write something like...

> "*The sequence is divergent to* ∞ *so*
>
> $$\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies a_n \geq M$$
>
> "*Take* $A = \min\{a_0, a_1, a_2, \ldots, a_{n_0-1}, M\}$"

... then your proof is incorrect. You need to fix a value of M first. Otherwise your variable M is quantified – it is a dummy variable, and it does not mean anything. Moreover, the value of n_0 depends on M. The above set does not make sense (or is not finite) unless we fix one single value of M and as a consequence one single value of n_0.

Q2 - Solution

As per the suggestion, watch the proof in Video 11.5. This one is very similar.

- Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Assume it is divergent to ∞. I want to prove that it is bounded below. In other words, I want to prove that

$$\exists A \in \mathbb{R},\ \forall n \in \mathbb{N},\ A \leq a_n$$

- By assumption $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞. Using 42 as the bound in the definition of "divergent to ∞", I know $\exists n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \quad n \geq n_0 \implies a_n \geq 42 \tag{1}$$

  Notice that I chose 42 at random. I could have chosen any other number as the bound. The point is to fix one.

- Next I choose

$$A = \min\{a_0, a_1, a_2, \ldots, a_{n_0-1}, 42\} \tag{2}$$

  I will now prove that this value of A satisfies

$$\forall n \in \mathbb{N},\ A \leq a_n.$$

- Let $n \in \mathbb{N}$. There are two cases to consider:
  - If $n < n_0$, then $A \leq a_n$ by the way we defined A in (2)
  - If $n \geq n_0$, then $a_n \geq 42$ from (1). Therefore $A \leq 42 \leq a_n$

  Either way, I have concluded that $A \leq a_n$, which is what I wanted to show.

□