MAT 137Y: Calculus with proofs
Test 1 - Part B - Version α

1. (a) Let g be a function with domain $\mathbb{R}$. Write the formal definition of $\lim_{x\to\infty} g(x) = \infty$.

   (b) Prove that
$$\lim_{x\to\infty} \left(\sqrt[3]{x} - 5\right) = \infty.$$
   Write a proof directly from the formal definition of limit. Do not use any of the limit laws or any other theorems.

Solutions:

(a) The standard definition is
$$\forall M \in \mathbb{R},\ \exists K \in \mathbb{R},\ \text{ such that } \ (\forall x \in \mathbb{R}) \quad x > K \implies g(x) > M$$
Notice that "$\forall x \in \mathbb{R}$" is implicit, and you do not need to write it explicitly.
In addition, you can substitute

- "$\forall M > 0$" for "$\forall M \in \mathbb{R}$"
- "$\exists K > 0$" for "$\exists K \in \mathbb{R}$"
- "$x \geq K$" for "$x > K$"
- "$g(x) \geq M$" for "$g(x) > M$"

(b) I want to show that
$$\forall M \in \mathbb{R},\ \exists K \in \mathbb{R},\ \text{ such that } \quad x > K \implies \sqrt[3]{x} - 5 > M$$

- Let us fix an arbitrary $M \in \mathbb{R}$.
- I take $K = (M+5)^3$
- Let $x \in \mathbb{R}$. Assume $x > K$. I want to show that $\sqrt[3]{x} - 5 > M$. The rest is algebra:

$$\begin{aligned} x &> K = (M+5)^3 \\ \sqrt[3]{x} &> M + 5 \\ \sqrt[3]{x} - 5 &> M \end{aligned}$$

□

2. Let $a \in \mathbb{R}$. Let f be a function with domain $\mathbb{R}$.

   (a) Write the "$\varepsilon - \delta$ version" of the definition of "f is continuous at a".

   (b) Assume f is continuous. Assume that $f(a) > 4$. Prove that there exists an open interval I, centered at a, such that
   $$\forall x \in I, \ f(x) > 4.$$
   Write a proof directly from the above definition of continuity. Do not use any of the limit laws or any other theorems.

Solutions:

(a) The standard definition is

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \text{ such that } \quad |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

There are a few possible variations:

- You can add "$\forall x \in \mathbb{R}$" right before the implication, but it is not needed: it is implicit.
- You can write "$0 < |x - a| < \delta$" instead of "$|x - a| < \delta$". Inside this statement, it does not change the meaning.

(b)

- Let us take $\varepsilon = f(a) - 4$. By hypothesis, $\varepsilon > 0$. I use this value of "ε" in the definition of "f is continuous at a", and I get that:

  $$\exists \delta > 0 \ \text{ such that } \quad |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

  I keep this value of δ.
- Then I define the interval I to be $I = (a - \delta, a + \delta)$.
  I will show that this interval satisfies

  $$\forall x \in I, f(x) > 4.$$

- Let $x \in I$. This is equivalent to $|x - a| < \delta$.
  Therefore, it follows that $|f(x) - f(a)| < \varepsilon$.
  In particular
  $$f(x) > f(a) - \varepsilon = f(a) - (f(a) - 4) = 4,$$
  which is what I wanted to show. □