If your high-school trigonometry consisted of a lot of memorization, your teacher was playing a prank on you. Let’s fix that.

The definition

- Draw the unit circle.
- Draw a radius from the origin such that the angle from the positive horizontal axis to this radius, measured counterclockwise, is \( x \).
- Find the point where this radius intersects the unit circle.
- The coordinates of this point are \((\cos x, \sin x)\).

The above is the definition of sin and cos. Forget all that nonsense about sohcahtoa.

The other trig functions are defined by

\[
\tan x = \frac{\sin x}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}.
\]
Basic properties

1. Using only the definition of \( \sin x \) and \( \cos x \) from the unit circle, figure out in which quadrants each of \( \sin \) and \( \cos \) are positive or negative.

2. Using only pictures from the definition of \( \sin \) and \( \cos \) from the unit circle, find formulas for the following, in terms of \( \sin x \) and \( \cos x \):
   
   (a) \( \sin(-x) \)
   (b) \( \cos(-x) \)
   (c) \( \sin(\pi/2 - x) \)
   (d) \( \cos(\pi/2 - x) \)
   (e) \( \cos(\pi/2 + x) \)
   (f) \( \sin(\pi + x) \)

3. Notice that the Pythagorean identity \( \sin^2 x + \cos^2 x = 1 \) is a direct consequence of the unit-circle definition.
   
   (a) Divide both sides by an appropriate quantity to get an identity involving \( \tan x \) and \( \sec x \).
   (b) Divide both sides by an appropriate quantity to get an identity involving \( \cot x \) and \( \csc x \).

4. You know there are rules for calculating the \( \sin \), \( \cos \), and \( \tan \) of an acute angle of a right triangle in terms of the lengths of the adjacent side, the opposite side, and the hypotenuse. See that they are just a consequence of the unit-circle definition by using congruent triangles.

Special values

5. The triangle below is a right triangle where the two equal sides have length 1. Use it to find the value of \( \sin(\pi/4) \), \( \cos(\pi/4) \), and \( \tan(\pi/4) \).

![Right Triangle](image)

6. The triangle \( \text{ABC} \) below is equilateral with side length 1. Use the triangle \( \text{ACD} \) to find the value of \( \sin(\pi/3) \), \( \cos(\pi/3) \), and \( \sin(\pi/6) \), and \( \cos(\pi/6) \).
7. Combine various of the previous questions to compute
   
   (a) $\sin\left(\frac{5\pi}{6}\right)$.
   (b) $\cos\left(\frac{3\pi}{2}\right)$.
   (c) $\tan\left(-\frac{\pi}{3}\right)$.

**Trig identities**

8. Memorize the identity
   
   $$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

   Now use your answers to questions 2a–2d to get formulas for:
   
   (a) $\cos(a - b)$
   (b) $\sin(a + b)$
   (c) $\sin(a - b)$

9. Use your answers to the previous question to get formulas for $\tan(a + b)$ and $\tan(a - b)$ in terms of $\tan a$ and $\tan b$.

10. Obtain formulas for $\cos(2x)$ and $\sin(2x)$ in terms of trig identities of the angle $x$.
    
    *Hint:* Use Question 8 with clever choices of $a$ and $b$.

11. Obtain formulas for $\cos x$ and $\sin x$ in terms of trig identities of the angle $2x$.
    
    *Hint:* One of the identities you have contains $\cos^2 x$ and $\sin^2 x$. Use it in combination with the Pythagorean identity.
12. *(This identity is useful when studying waves in physics.)*

Write the quantity \( \sin x + \sin y \) as a product of various sines and/or cosines.

*Hint:* Stare at the identities for \( \sin(a + b) \) and \( \sin(a - b) \).

13. *(This is the inverse problem to the previous one. It is useful for integration.)*

Write the following quantities

(a) \( \sin(5x) \sin(3x) \),

(b) \( \sin(2x) \cos x \).

as a linear combination of single sines and/or cosines with constant coefficient.

*Hint:* Think of two different identities which both contain something like \( \sin a \sin b \).