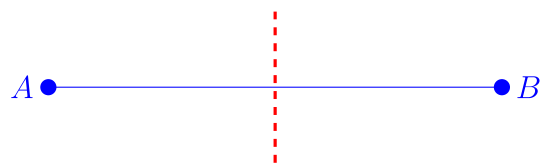
MAT 137Y: Calculus with proofs
Assignment 5 - Sample solutions

1. Every morning Neo packs his backpack and walks a distance L through a straight path in the forest from his home (A) to the unicorn sanctuary (B). One day he discovers someone has built an electric fence in the exact middle of his daily path (the dashed, red line in the picture):

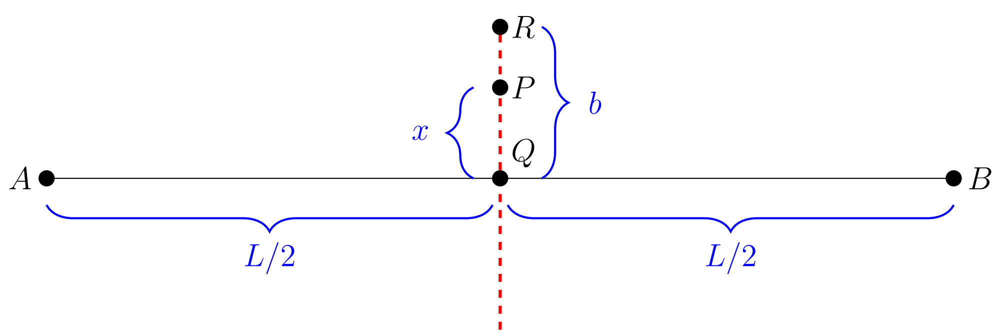


The fence has length $2b$ – it extends for a distance b on each side of the path – and is perpendicular to the path. After a few days, Neo notices that the electric fence is turned on only half the time, but he does not know if the fence is on or off on any given day until he walks up to it and throws his cat at the fence to test it. If the fence is off, he can just quickly climb over it. Otherwise, he has to walk around it. He devises a plan: he will walk straight from his home to some point P in the fence; then, he will walk around it or climb over it depending on whether the fence is on or off. Which point P should he choose in order to minimize the *average* length of his trip?

Solution:

PART 1: MODELLING

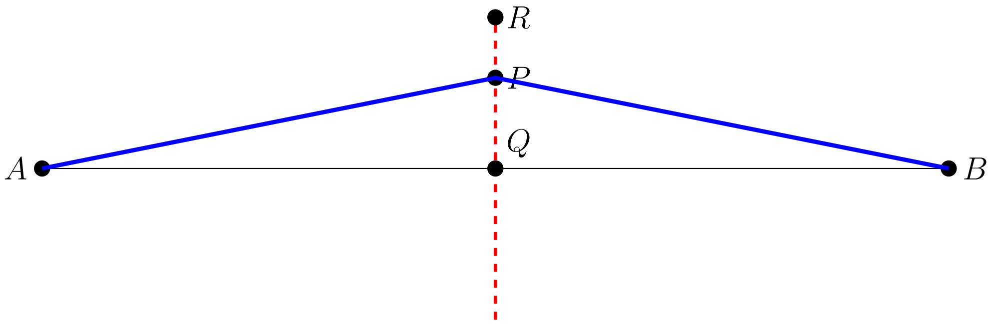
Let us add a few more labels to the picture:



P is a generic point in the fence, which represents the point Neo may choose on the fence (he initially walks from A to P) . Let us call x the distance between P and Q. It will be our variable. I have also marked the distances b and $L/2$ on the picture; they are constant. Notice that $0 \leq x \leq b$.

For convenience, in what follows, when I write $|XY|$, it represents the distance between points X and Y.

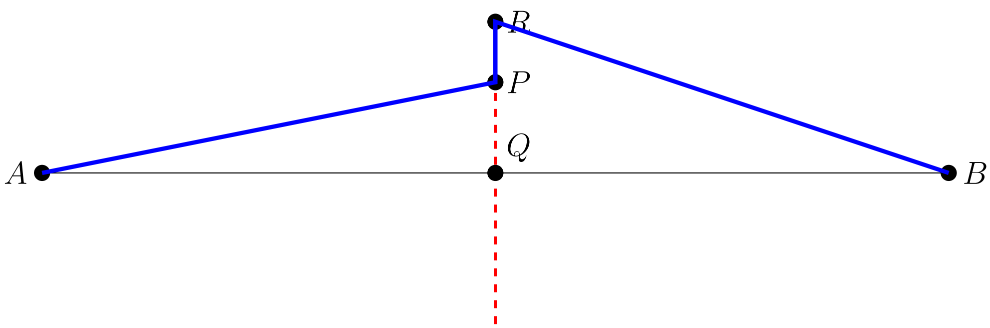
When the fence is off, Neo's path will go from A to P to B:



and the distance will be

$$g(x) \;=\; |AP| + |PB| \;=\; \sqrt{(L/2)^2 + x^2} + \sqrt{(L/2)^2 + x^2} \;=\; 2\sqrt{(L/2)^2 + x^2}$$

When the fence is on, Neo's path will go from A to P to R to B:



and the distance will be

$$h(x) \;=\; |AP| + |PR| + |RB| \;=\; \sqrt{(L/2)^2 + x^2} + (b - x) + \sqrt{(L/2)^2 + b^2}$$

The average of the two lengths is $f(x) = \dfrac{1}{2}\,[g(x) + h(x)]$. After simplifying

$$f(x) \;=\; \frac{3}{2}\sqrt{(L/2)^2 + x^2} - \frac{x}{2} + \text{constants}, \qquad 0 \le x \le b.$$

This is the function we want to mimize.

PART 2: CALCULUS

The function f is continuous on the interval $[0, b]$. By the EVT, it must have a minimum. The minimum must occur at an endpoint of the interval or at a critical point. Since f is differentiable on the domain, the only critical points are values of x where $f'(x) = 0$. Using the standard differentiation rules:

$$f'(x) \;=\; \frac{3}{2}\frac{x}{\sqrt{(L/2)^2 + x^2}} - \frac{1}{2} \tag{1}$$

To find the critical points, we need to solve the equation

$$3x - \sqrt{(L/2)^2 + x^2} \;=\; 0$$

After some algebra there are two solutions

$$x = \pm\frac{L}{4\sqrt{2}}$$

The negative solution is clearly outside of the domain. Let us call the positive solution $\boxed{x_0 = \frac{L}{4\sqrt{2}}}$. Notice that x_0 may or may not be in the domain, depending on b.

- **Case 1:** When $x_0 < b$. In this case x_0 is an interior point to the domain and it is a critical point. Moreover, from Equation (1) we can solve that $f'(x) < 0$ for $x \in [0, x_0)$ and $f'(x) > 0$ for $x \in (x_0, b]$. This means that f has an absolute minimum at x_0 and that is the point that Neo should choose.
- **Case 2:** When $x_0 \geq b$. In this case there are no critical points. The minimum must happen at one of the two endpoints. We could now evaluate $f(0)$ and $f(b)$ to compare them, but there is a faster way. Look at the equation for f' in (1). We see that f' is continuous on the domain. We also know that f' is never 0 on the domain. Therefore f' must remain always positive or always negative. Since $f'(0) < 0$, f' is always negative, and f is decreasing on the domain. Thus, the minimum happens at $x = b$. Neo should walk directly to the end of the fence and not bother.

CONCLUSION

If $x_0 < b$, then Neo should walk to the point P defined by $x = x_0$. Otherwise, Neo should walk directly from home to the end of the fence.

2. (a) Let f be a function with domain $\mathbb{R}$. Assume f has derivatives of every order. Find all possible real numbers $A, B, C \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{f(x) - [Ax^2 + Bx + C]}{x^2} = 0. \tag{2}$$

Note: In your answer, A, B and C will depend on values of f and its derivatives. We are asking for *all* possible answers. We want you to prove that your choices of A, B, and C satisfy (2), and that there are no other choices that satisfy (2).

Solution: The only choice that satisfies (2) is

$$A = \frac{1}{2} f''(0), \quad B = f'(0), \quad C = f(0). \tag{3}$$

I will prove *simultaneously* that the choices in (3) work and that no other choices work.

We are assuming that f has derivatives of every order. Therefore f and all its derivatives are continuous (because they are differentiable).

To begin, let us fix arbitrary real numbers A, B, and C, and let us define the limit (if it exists)

$$L_1 = \lim_{x\to 0} \frac{f(x) - [Ax^2 + Bx + C]}{x^2} \tag{4}$$

Since f is continuous, the limit of the numerator in (4) is $f(0) - C$.

- If $C \neq f(0)$, the limit L_1 in (4) is of the form $\dfrac{\text{non-zero number}}{0}$. Notice also that the denominator is always positive (except at $x = 0$). Therefore, L_1 is either ∞ or $-\infty$, depending on the sign of $f(0) - C$. Either way, the limit L_1 is not 0 in this case.

For the rest of this solution, I will explore what happens when $C = f(0)$. In that case, we have an indeterminate form of type $0/0$ in (4). We can attempt to use L'Hôpital's Rule. After using it, we get the new limit:

$$L_2 = \lim_{x\to 0} \frac{f'(x) - [2Ax + B]}{2x} \tag{5}$$

Since f' is continuous, the limit of the numerator in (5) is $f'(0) - B$.

- If $B \neq f'(0)$, the limit L_2 in (5) is of the form $\dfrac{\text{non-zero number}}{0}$. In addition, the denominator is postive as $x \to 0^+$ and negative as $x \to 0^-$. Therefore, each of the two side limits in (5) is ∞ or $-\infty$ (and they have opposite signs). In any case, using L'Hôpital's Rule was legal (for the left- and right-side limits separately) and hence "$L_1 = L_2$". Once again, the limit L_1 is not 0 in this case.

For the rest of this solution, I will explore what happens when $B = f'(0)$. In that case, we have an indeterminate form of type $0/0$ in (5). We can attempt to use L'Hôpital's Rule again. After using it, we get the new limit:

$$L_3 = \lim_{x\to 0} \frac{f''(x) - [2A]}{2}$$

Since f'' is continuous, we get that

$$L_3 \;=\; \frac{f''(0) - 2A}{2}$$

This limit exists! Therefore, both uses of L'Hôpital were legal, and we get

$$L_1 \;=\; L_2 \;=\; L_3 \;=\; \frac{f''(0) - 2A}{2}$$

Therefore

- If $A \neq \frac{1}{2} f''(0)$ the limit L_1 exists but is not 0.
- If $A = \frac{1}{2} f''(0)$, I finally have $L_1 = 0$.

Reviewing all the things I have found, I have proven that the choices in (3) work and that no other choices work.

(b) Let f be a function with domain $\mathbb{R}$. Assume f has derivatives of every order. Let N be a positive integer. Find a polynomial P_N such that

$$\lim_{x\to 0} \frac{f(x) - P_N(x)}{x^N} = 0$$

Suggestion: You may want to do some rough work until you can form a conjecture. Do not submit the rough work. To prove your conjecture, use induction.

Solution:

For every function f that has derivatives of all orders, and for every positive integer N, I define the polynomial

$$\boxed{P_{N,f}(x) = \frac{f^{(N)}(0)}{N!}x^N + \frac{f^{(N-1)}(0)}{(N-1)!}x^{N-1} + \ldots + \frac{f'''(0)}{3!}x^3 + \frac{f''(0)}{2}x^2 + f'(0)x + f(0)} \tag{6}$$

I am going to show that

$$\lim_{x\to 0} \frac{f(x) - P_{N,f}(x)}{x^N} = 0 \tag{7}$$

Specifically, for every positive integer N, I define the statement

S_N = "For all functions f that have domain $\mathbb{R}$ and derivatives of all order, (7) holds."

I am going to prove this statement by induction on N.

BASE CASE ($N = 1$).[1]

Let f be a function that has derivatives of all order. I need to show that

$$\lim_{x\to 0} \frac{f(x) - [f(0) - f'(0)x]}{x} = 0$$

Indeed

$$\lim_{x\to 0} \frac{f(x) - [f(0) - f'(0)x]}{x} = \lim_{x\to 0} \left[\frac{f(x) - f(0)}{x} - f'(0)\right] = f'(0) - f'(0) = 0,$$

where I have used the definition of the derivative at 0 as a limit.

[1] It is also possible to prove the claim for all natural numbers N, and then the base case would be $N = 0$. However, since the question asked for positive integers N, I will start at $N = 1$.

INDUCTION STEP.

- Let us fix a positive integer N. I will assume the statement S_N. In other words, I assume that (7) holds for *all* functions that have derivatives of all orders. I want to prove the statement S_{N+1}. To do so, let us fix a function f that has derivatives of all orders. I want to prove that, for this function f, the limit

$$L_1 \;=\; \lim_{x\to 0} \frac{f(x) - P_{N+1,f}(x)}{x^{N+1}} \tag{8}$$

is 0.

- Since f is continuous, the limit of the numerator in (8) is

$$f(0) - P_{N+1,f}(0) \;=\; f(0) - f(0) = 0.$$

Therefore, we have an indeterminate form of type 0/0 in (8) and we can try to use L'Hôpital's Rule. After using L'Hôpital's Rule we get the limit

$$L_2 \;=\; \lim_{x\to 0} \frac{f'(x) - P'_{N+1,f}(x)}{(N+1)x^N} \tag{9}$$

- I am going to try to use the induction hypothesis to conclude that the limit in (9) is 0. Since f has derivatives of all orders, so does f'. Therefore the induction hypothesis also applies to f', and, specifically, we know that

$$L_3 \;=\; \lim_{x\to 0} \frac{f'(x) - P_{N,f'}(x)}{x^N}$$

- What is the relation between $P'_{N+1,f}$ and $P_{N,f'}$? On the one hand, by definition

$$P_{N+1,f}(x) \;=\; \frac{f^{(N+1)}(0)}{(N+1)!}x^{N+1} + \frac{f^{(N)}(0)}{N!}x^N + \cdots + \frac{f'''(0)}{3!}x^3 + \frac{f''(0)}{2}x^2 + f'(0)x + f(0)$$

and so its derivative is

$$\begin{aligned}
\boxed{P'_{N+1,f}(x)} &= \frac{f^{(N+1)}(0)}{(N+1)!}(N+1)x^N + \frac{f^{(N)}(0)}{N!}Nx^{N-1} + \cdots + \frac{f'''(0)}{3!}3x^2 + \frac{f''(0)}{2!}2x + f'(0) \\
&= \frac{f^{(N+1)}(0)}{N!}x^N + \frac{f^{(N)}(0)}{(N-1)!}x^{N-1} + \cdots + \frac{f'''(0)}{2!}x^2 + f''(0)x + f'(0)
\end{aligned}$$

On the other hand

$$\boxed{P_{N,f'}(x)} \;=\; \frac{f^{(N+1)}(0)}{N!}x^N + \frac{f^{(N)}(0)}{(N-1)!}x^{N-1} + \cdots + \frac{f'''(0)}{2!}x^2 + f''(0)x + f'(0)$$

Therefore

$$P'_{N+1,f} = P_{N,f'}.$$

- Now let's put it all together:

$$\begin{aligned} L_2 &= \lim_{x\to 0} \frac{f'(x) - P'_{N+1,f}(x)}{(N+1)x^N} \\ &= \lim_{x\to 0} \left[\frac{1}{N+1} \cdot \frac{f'(x) - P_{N,f'}(x)}{x^N} \right] \;=\; \frac{1}{N+1} \cdot L_3 \;=\; \frac{1}{N+1} \cdot 0 \;=\; 0 \end{aligned}$$

Therefore, the limit L_2 in (9) is 0, the use of L'Hôpital's Rule was acceptable, and the limit L_1 in (8) is 0 as well. This is what we wanted to prove. We are done!

(c) Using your new result, find polynomials P and Q such that

$$\lim_{x\to 0} \frac{e^x - P(x)}{x^6} = 0, \qquad \lim_{x\to 0} \frac{\sin x - Q(x)}{x^{11}} = 0.$$

Solution:

We can use the result from part (b): Equation (6).

- We notice that we can choose $P = P_{6,f}$ as defined by (6), for the function $f(x) = e^x$. Therefore:
- We notice that $P = P_{6,f}$ for the function $f(x) = e^x$. Therefore:

$$P(x) = \frac{f^{(6)}(0)}{6!}x^6 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f'''(0)}{3!}x^3 + \frac{f''(0)}{2!}x^2 + f'(0)x + f(0)$$

The exponential function satisfies that $f^{(n)}(x) = e^x$, and hence $f^{(n)}(0) = 1$, for all $n \in \mathbb{N}$. Therefore:

$$P(x) = \frac{1}{6!}x^6 + \frac{1}{5!}x^5 + \frac{1}{4!}x^4 + \frac{1}{3!}x^3 + \frac{1}{2!}x^2 + x + 1$$

- Similarly, we notice that we can choose $Q = P_{11,g}$ as defined by (6), for the function $g(x) = \sin x$. Therefore:

$$Q(x) = \frac{g^{(11)}(0)}{11!}x^{11} + \frac{g^{(10)}(0)}{10!}x^{10} + \ldots + \frac{g''(0)}{2!}x^2 + g'(0)x + g(0)$$

The sine function satisfies that

$$g'(x) = \cos x, \quad g''(x) = -\sin x, \quad g'''(x) = -\cos x, \quad g^{(4)}(x) = \sin x, \ldots$$

and after that the derivatives cycle. Thus, the even derivatives at 0 are 0, and the odd derivatives alternate between 1 and -1. Therefore

$$Q(x) = -\frac{1}{11!}x^{11} + \frac{1}{9!}x^9 - \frac{1}{7!}x^7 + \frac{1}{5!}x^5 - \frac{1}{3!}x^3 + x.$$

3. In Video 6.13, you learned about various geometrical notions that we could have used to define concavity. Here is yet another one.

   Let f be a function defined on an interval I. Given two points P and Q on the graph of f, we will call $m_{P,Q}$ the slope of the line going through P and Q. We say that the function f is "cave up" on I when for every 3 different points P, Q, and R on the graph of f, if P is to the left of Q, and Q is to the left of R, then $m_{P,Q} < m_{Q,R}$. Sketch a graph and make sure you understand this definition geometrically before continuing.

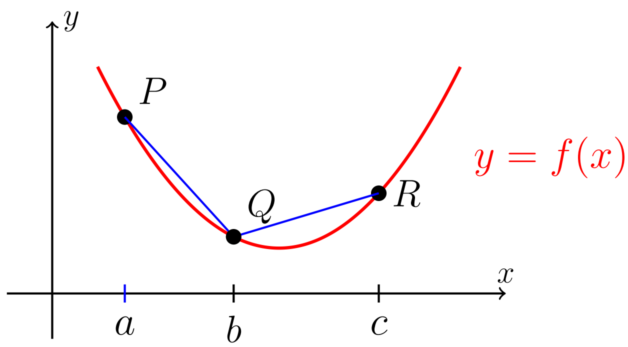
   Assume f is differentiable on I. Prove that IF f is concave up on I, THEN f is cave up on I.

   Hint: Use MVT.

   Note: It is also possible to prove that cave up implies concave up, but we will skip it for now. In fact, all of the different versions of concavity you have learned are equivalent for differentiable functions.

Solution:

- Assume that f is concave up on I. I want to prove that f is cave up on I. Let us fix three points P, Q, R on the graph of f such that P is to the left of Q, and Q is to the left of R. This means that we can find $a, b, c \in I$ such that $a < b < c$ and $P = (a, f(a)), Q = (b, f(b)), R = (c, f(c))$.



  The slopes of the secant lines are given by

$$m_{P,Q} = \frac{f(b) - f(a)}{b - a}, \qquad m_{Q,R} = \frac{f(c) - f(b)}{c - b}.$$

  I need to prove that $m_{P,Q} < m_{Q,R}$.

- The function f is differentiable (and thus continuous) on I, hence f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, there exists $m \in (a, b)$ such that

$$f'(m) = \frac{f(b) - f(a)}{b - a}.$$

  Similarly, repeating the same argument for f on $[b, c]$, we find that there exists $n \in (b, c)$ such that

$$f'(n) = \frac{f(c) - f(b)}{c - b}.$$

  To conclude the proof I need to show that $f'(m) < f'(n)$

- Notice that

  $$a < m < b < n < c,$$

  and thus $m < n$. Since f is concave up on I, we know that f' is increasing on I, and therefore $f'(m) < f'(n)$. That is what I needed to show. Yay!

4. Let's recall the definition of horizontal/slant asymptote. Let f be a function defined at least on an interval (c, ∞) for some $c \in \mathbb{R}$. We say that f has an asymptote as $x \to \infty$ when there exist numbers $m, b \in \mathbb{R}$ such that

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0. \tag{10}$$

Notice that this includes both slant asymptotes (when $m \neq 0$) and horizontal asymptotes (when $m = 0$).

Consider the following two claims:

Claim A: IF f has an asymptote as $x \to \infty$, THEN $\lim_{x\to\infty} \frac{f(x)}{x}$ exists.

Claim B: IF $\lim_{x\to\infty} \frac{f(x)}{x}$ exists, THEN f has an asymptote as $x \to \infty$.

(a) Prove that Claim A is true.

Solution:

Assume that f has an asymptote as $x \to \infty$. This means there exist $m, b \in \mathbb{R}$ satisfying (10). In this case

$$\lim_{x\to\infty} \frac{f(x) - (mx + b)}{x} = 0$$

because the numerator has limit 0 and the denominator has limit ∞. Then

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{[f(x) - (mx + b) + mx + b]}{x}$$
$$= \lim_{x\to\infty} \left[\frac{f(x) - (mx + b)}{x} + m + \frac{b}{x}\right] = 0 + m + 0 = m,$$

where I have used the limit law for sums. Hence $\lim_{x\to\infty} \frac{f(x)}{x}$ exists (and equals m).

(b) Prove that Claim B is false.

Solution: I will prove that $f(x) = \sin x$ is a counterexample.[2] Specifically, I will prove that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$, but sin has no asymptote as $x \to \infty$.

- **Claim 1:** $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ (and hence exists).

  This is due to the Squeeze Theorem, since

  $$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

  for all $x > 0$ and

  $$\lim_{x\to\infty} \frac{-1}{x} \;=\; \lim_{x\to\infty} \frac{1}{x} \;=\; 0.$$

- **Claim 2:** sin does not have an asymptote as $x \to \infty$.

  I need to prove that for every $m, b \in \mathbb{R}$

  $$\lim_{x\to\infty} [\sin x - (mx + b)] \;\neq\; 0$$

  This is a proof by cases.

  - If $m \neq 0$, notice that

    $$\lim_{x\to\infty} \left[\frac{\sin x}{x} - m + \frac{b}{x}\right] \;=\; 0 - m + 0 \;=\; -m$$

    Therefore

    $$\lim_{x\to\infty} [\sin x - (mx + b)] = \lim_{x\to\infty} x \cdot \left[\frac{\sin x}{x} - m + \frac{b}{x}\right] \;=\; \pm\infty$$

    where the limit is ∞ if $m < 0$ and $-\infty$ if $m > 0$. Either way the limit does not exist.
  - On the other hand, if $m = 0$, we get

    $$\lim_{x\to\infty} [\sin x - b]$$

    which does not exist, because sin oscillates indefinitely between -1 and 1 as $x \to \infty$.

  Note: There is a way to skip one case for the proof of Claim 2. In the proof of Claim A we had noticed that if $y = mx + b$ is an asymptote for f as $x \to \infty$, then necessarily $m = \lim_{x\to\infty} \frac{f(x)}{x}$, which in this case is 0. So when verifying that sin does not have an asymptote, it would have been enough to check that $y = b$ is not an asymptote, rather than the more general form $y = mx + b$.

[2] There are many other counterexamples, such as $f(x) = x + \sin x$, $f(x) = \sqrt{x}$, or $f(x) = \ln x$.

(c) Here is one more false claim and a bad proof.

Claim C: Assume the function f is differentiable and that $\lim_{x\to\infty} f(x) = \infty$.

$$\lim_{x\to\infty} \frac{f(x)}{x} \text{ exists} \quad \iff \quad \lim_{x\to\infty} f'(x) \text{ exists}$$

"Proof": We can use L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{f'(x)}{1} = \lim_{x\to\infty} f'(x)$$

□

Explain the error in the proof.

Then prove that the claim is false with a counterexample.

Solution:

We do not get an "if an only if".

- IF $\lim_{x\to\infty} f'(x)$ exists, then we can use L'Hôpital's Rule, and we conclude that

  $$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f'(x)$$

  also exists. This part was correct.
- However, IF $\lim_{x\to\infty} f'(x)$ does not exist (and is not $\pm\infty$ either), then L'Hôpital's Rule does not apply (that is the error in the proof!) and it is still possible that

  $$\lim_{x\to\infty} \frac{f(x)}{x}$$

  exists.

As a counterexample, consider $f(x) = x + \sin x$. Then

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x + \sin x}{x} = \lim_{x\to\infty} \left[1 + \frac{\sin x}{x}\right] = 1 + 0 = 0$$

However,

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} [1 + \cos x]$$

which does not exist.