MAT 137Y: Calculus with proofs
Assignment 2 - Sample solutions

Question 1 Sketch the graph of a function $h$ that satisfies all the following properties at once:

(a) The domain of $h$ is $\mathbb{R}$.

(b) $\lim_{x \to 2} h(x) = 0$ and $\lim_{x \to 2} h(h(x)) = \infty$.

(c) $\lim_{x \to 4} h(x) = 0$ and $\lim_{x \to 4} h(h(x)) = -\infty$.

(d) $\lim_{x \to 0} h(h(x)) = 3$.

(e) $\lim_{x \to -3^+} h(x) = 0$ and $\lim_{x \to -3^+} h(h(x))$ does not exist, is not $\infty$, and is not $-\infty$.

Solution: The graph is sketched in Figure 1.

![Graph of function h](image)

Figure 1: The function $h$ oscillates infinitely many times between positive and negative values when $x$ approaches $-3$ from the right, but the amplitude of these oscillations approaches $0$ as $x \to -3^+$.

(a) The function is defined for all real numbers, so the domain is $\mathbb{R}$. 
(b) When $x$ approaches 2, the value of $h(x)$ approaches 0 while remaining negative; moreover, for negative values of $x$ approaching 0, $h(x)$ approaches $\infty$. Therefore

$$\lim_{x \to 2} h(x) = 0, \quad \lim_{x \to 2} h(h(x)) = \lim_{x \to 0^-} h(x) = \infty.$$  \hfill (1)

(c) When $x$ approaches 4, the value of $h(x)$ approaches 0 while remaining positive; moreover, for positive values of $x$ approaching 0, $h(x)$ approaches $-\infty$. Therefore

$$\lim_{x \to 4} h(x) = 0, \quad \lim_{x \to 4} h(h(x)) = \lim_{x \to 0^+} h(x) = -\infty.$$  \hfill (2)

(d) When $x$ approaches $\infty$ or $-\infty$, $h(x)$ approaches 3. I may then write

$$\lim_{x \to 0^-} h(h(x)) = \lim_{x \to \infty} h(x) = 3 \quad \text{and} \quad \lim_{x \to 0^+} h(x) = \lim_{x \to -\infty} h(x) = 3.$$  \hfill (3)

Since the two side limits of $h(h(x))$ for $x$ approaching 0 exist and are both equal to 3, I can conclude that $\lim_{x \to 0} h(h(x)) = 3$.

(e) When $x$ approaches $-3$ (and $x > 3$), $h(x)$ has infinitely many oscillations; however the graph shows that its values are ”squeezed” to 0, or in other words $\lim_{x \to -3^+} h(x) = 0$. On the other hand, since $h(x)$ oscillates between small positive and negative numbers, $h(h(x))$ swings between positive and negative numbers, arbitrarily large in absolute value. Therefore, $h(h(x))$ does not approach any real number $L$, and the limit does not exist. The limit is not $\infty$ nor $-\infty$, either, because $h(h(x))$ keeps switching between positive and negative values, rather than growing arbitrarily large in either direction.
Question 2 Let $a \in \mathbb{R}$. Let $f$ and $g$ be two functions that are defined, at least, on an interval centered at $a$, except maybe at $a$. Assume that \( \lim_{x \to a} f(x) \) does not exist, and that \( \lim_{x \to a} g(x) \) does not exist. Based only on this information, can you conclude whether \( \lim_{x \to a} [f(x) + g(x)] \) exists or does not exist? Prove it.

Solution:
No, we cannot conclude whether exists or does not exist based on this information.
To prove this, I will give two examples of $a$, $f$, and $g$ as above, so that the limit exists in one case but not in the other.

- Example 1: Let $a = 0$ and $f(x) = g(x) = \frac{1}{x}$.
  \[ \lim_{x \to 0} f(x) \text{ DNE}, \quad \lim_{x \to 0} g(x) \text{ DNE}, \quad \text{and} \quad \lim_{x \to 0} (f(x) + g(x)) \text{ DNE.} \]

- Example 2: Let $a = 0$, $f(x) = \frac{1}{x}$, and $g(x) = -\frac{1}{x}$.
  \[ \lim_{x \to 0} f(x) \text{ DNE}, \quad \lim_{x \to 0} g(x) \text{ DNE,} \quad \text{but} \quad \lim_{x \to 0} (f(x) + g(x)) = \lim_{x \to 0} 0 = 0. \]
Question 3 Prove that \( \lim_{x \to 2} x^3 = 8 \). Write a proof directly from the definition of limit, without using any of the limit laws or other theorems.

Proof:
I want to show that

\[
\forall \varepsilon > 0, \exists \delta > 0, \quad 0 < |x - 2| < \delta \implies |x^3 - 8| < \varepsilon
\]

- Fix \( \varepsilon > 0 \).
- Let \( \delta = \min \left\{ 1, \frac{\varepsilon}{20} \right\} \).
- Let \( x \in \mathbb{R} \). Assume \( 0 < |x - 2| < \delta \). I will show that \( |x^3 - 8| < \varepsilon \).
- I can draw the following conclusions:
  - Since \( \delta \leq \frac{\varepsilon}{20} \) and \( |x - 2| < \delta \), I have that \( |x - 2| < \frac{\varepsilon}{20} \).
  - Since \( \delta \leq 1 \) and \( 2 - \delta < x < 2 + \delta \), it follows that \( 1 < x < 3 \).
    Combining with the triangular inequality:
    \[
    |x^2 + 2x + 4| \leq x^2 + 2x + 4 < 9 + 6 + 4 = 19.
    \]
- The two inequalities above imply that
  \[
  |x^3 - 8| = |x - 2| |x^2 + 2x + 4| < \frac{\varepsilon}{20} \cdot 19 < \varepsilon.
  \]

This is what I needed to prove.

\( \square \)
Question 4 Let $f$ and $g$ be two functions with domain $\mathbb{R}$. Let $h = f + g$. Prove that

$$\text{IF } \lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x) \text{ exists,}$$

$$\text{THEN } \lim_{x \to \infty} h(x) = \infty.$$ 

Write a proof directly from the definition of limit, without using any of the limit laws or other theorems.

Proof:

- I want to prove that $\forall M \in \mathbb{R}, \exists N \in \mathbb{R}$, such that $x > N \implies h(x) > M$

- Fix $M \in \mathbb{R}$. Call $L = \lim_{x \to \infty} g(x)$.

  - Use "$\varepsilon = 1$" in the definition of $\lim_{x \to \infty} g(x) = L$:
    $$\exists N_2 > 0 \text{ such that } x > N_2 \implies |g(x) - L| < 1$$

  - Use "$M_1 = M - L + 1$" as the cut-off in the definition of $\lim_{x \to \infty} f(x) = \infty$:
    $$\exists N_1 > 0 \text{ such that } x > N_1 \implies f(x) > M - L + 1$$

  Take $N = \max\{N_1, N_2\}$

- Let $x \in \mathbb{R}$. Assume $x > N$. I will show that $h(x) > M$.

  I can conclude that
  - $x > N \geq N_1$ so $f(x) > M - L + 1$
  - $x > N \geq N_2$ so $|g(x) - L| < 1$, and in particular $g(x) > L - 1$

  Using both inequalities:
  $$h(x) = f(x) + g(x) > (M - L + 1) + (L - 1) = M$$

  which is what I had to prove.