MAT 137Y: Calculus!
Problem Set 7
Sample solutions

1. Let $a, b, c, k \in \mathbb{R}$. Compute the following limit

$$\lim_{x \to 0} \frac{\int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds \right] dt}{\cos x - 1}$$

We first introduce intermediate functions that will be useful to explain the computation.

- The function $f(x) = e^{-x^2}$ is continuous on $\mathbb{R}$. Hence, according to FTC–1, the function
$$F(x) = \int_0^x e^{-s^2} ds$$
is well-defined and differentiable on $\mathbb{R}$ and moreover $F'(x) = e^{-x^2}$.
- The function
$$g(x) = \int_{cx}^{kx} e^{-s^2} ds = \int_0^{kx} e^{-s^2} ds - \int_0^{cx} e^{-s^2} ds = F(kx) - F(cx)$$
is differentiable on $\mathbb{R}$ since F is.
- Since g is continuous, as a differentiable function, we can apply FTC–1 again to show that the function
$$G(x) = \int_0^x g(t) dt$$
is well-defined, differentiable on $\mathbb{R}$ and that $G'(x) = g(x)$.
- Set
$$H(x) = \int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds \right] dt = G(bx) - G(ax)$$
- We know that H is differentiable since G is.

Notice that
$$\frac{\int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds \right] dt}{\cos x - 1} = \frac{H(x)}{\cos(x) - 1}$$

We are going to compute the limit of the question by applying L'Hôpital's rule twice.

Step 1: Use L'Hôpital's Rule on $\lim_{x\to 0} \dfrac{H(x)}{\cos(x)-1}$.

Let's check all the hypotheses of L'Hôpital's Rule:

- By continuity of G and cos, we know that

$$\lim_{x\to 0} H(x) = G(0) - G(0) = 0$$

  and that

$$\lim_{x\to 0} \cos(x) - 1 = \cos(0) - 1 = 0$$

- According to the chain rule, H is differentiable.
- The denominator is differentiable and $\frac{d}{dx}\big(\cos(x)-1\big) = -\sin(x)$.
- There exists a small interval centered at 0 such that $\cos(x) - 1 \neq 0$ and $-\sin(x) \neq 0$ on this interval except at 0.
- Thus, IF we prove that $\lim_{x\to 0} \dfrac{H'(x)}{\frac{d}{dx}[\cos x - 1]}$ exists (we will conclude this soon), then we can use L'Hôpital and write

$$\lim_{x\to 0} \frac{H(x)}{\cos x - 1} = \lim_{x\to 0} \frac{H'(x)}{\frac{d}{dx}[\cos x - 1]} = \lim_{x\to 0} \frac{bG'(bx) - aG'(ax)}{-\sin(x)} = \lim_{x\to 0} \frac{ag(ax) - bg(bx)}{\sin(x)}$$

Step 2: Use L'Hôpital's Rule on $\lim_{x\to 0} \dfrac{ag(ax) - bg(bx)}{\sin(x)}$

Let's check all the hypotheses of L'Hôpital's Rule:

- Since g is continuous, we have

$$\lim_{x\to 0} \Big(ag(ax) - bg(bx)\Big) = ag(0) - bg(0) = 0 - 0 = 0$$

  and since sin is continuous, we have

$$\lim_{x\to 0} \sin(x) = \sin(0) = 0$$

- Using the differentiation rules (including the chain rule), we know that the numerator is differentiable.
- The denominator is differentiable and $\sin'(x) = \cos(x)$.
- There exists a small interval centered at 0 such that $\sin(x) \neq 0$ and $\sin'(x) = \cos(x) \neq 0$ on this interval, except at 0 for sin.

- Thus, IF we prove that $\lim_{x\to 0} \frac{\frac{d}{dx}\left[ag(ax) - bg(bx)\right]}{\frac{d}{dx}\left[\sin x\right]}$ exists (we will conclude this soon), then we can use L'Hôpital and write:

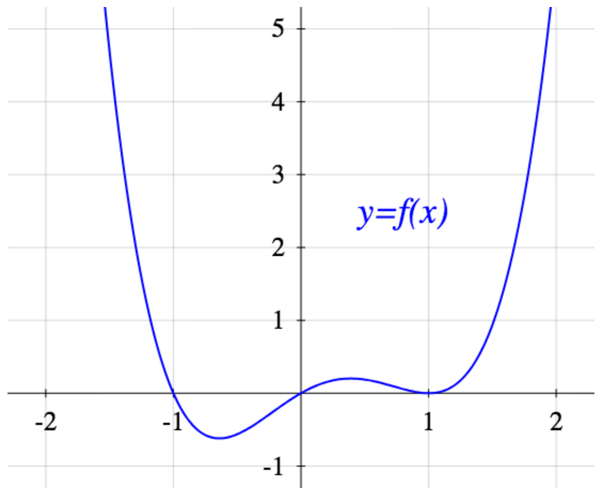
$$\begin{aligned}
\lim_{x\to 0} \frac{ag(ax) - bg(bx)}{\sin x} &= \lim_{x\to 0} \frac{\frac{d}{dx}\left[ag(ax) - bg(bx)\right]}{\frac{d}{dx}\left[\sin x\right]} = \lim_{x\to 0} \frac{a^2 g'(ax) - b^2 g'(bx)}{\cos x} \\
&= \lim_{x\to 0} \frac{a^2 k e^{-(kax)^2} - a^2 c e^{-(cax)^2} - b^2 k e^{-(kbx)^2} + b^2 c e^{-(cbx)^2}}{\cos(x)} \\
&= \frac{a^2 k - a^2 c - b^2 k + b^2 c}{1} = (a^2 - b^2)(k - c)
\end{aligned}$$

Since this limit exists, then the second use of L'Hôpital's Rule is valid, and therefore the first use of L'Hôpital's Rule is also valid.

<u>Conclusion:</u>

$$\lim_{x\to 0} \frac{\int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds\right] dt}{\cos x - 1} = (a^2 - b^2)(k - c)$$

2. Below is the graph of the function f:



The domain of f is $\mathbb{R}$ and the graph continues to the right and to the left as you expect. We define a new function H by

$$H(x) = \int_0^{\int_0^x f(t)dt} f(s)ds$$

How many local maxima and local minima does H have? Give the approximate x-coordinate for each one of them.

- We know that f is continuous on $\mathbb{R}$ hence, according to FTC–1,

$$F(x) = \int_0^x f(t)dt$$

  is well-defined, differentiable on $\mathbb{R}$ and moreover $F' = f$.

- Notice that

$$H(x) = \int_0^{\int_0^x f(t)dt} f(s)ds = \int_0^{F(x)} f(s)ds = F(F(x))$$

- Hence, by the chain rule, H is differentiable on $\mathbb{R}$ and

$$H'(x) = F'(x)F'(F(x)) = f(x)f(F(x))$$

- We know that the local extrema are reached at the critical points, i.e. at $x \in \mathbb{R}$ such that $H'(x) = 0$.

Since

$$H'(x) = 0 \Leftrightarrow \Big(f(x) = 0 \text{ or } f(F(x)) = 0 \Big)$$

we are going to study these two cases separately.

Case 1: $f(x) = 0$
According to the graph, there are three solutions which are -1, 0, and 1.

Case 2: $f(F(x)) = 0$
Using the previous case, we are looking for $x \in \mathbb{R}$ such that $F(x) = -1$, 0 or 1.

The graph of f allows us to write the following variation table:

x	$-\infty$		-1		0		1		$+\infty$
$F'(x) = f(x)$		$+$	0	$-$	0	$+$	0	$+$	
$F(x)$	$-\infty$	↗	α	↘	0	↗	β	↗	$+\infty$

Moreover, still using the graph of f, we know that

- $\alpha = F(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^{0} f(t)dt \in (0,1)$ (that's the yellow area below), i.e.

$$0 < \alpha < 1$$

- $\beta = F(1) = \int_0^1 f(t)dt \in (0,1)$ (that's the first part of the blue area below), i.e.

$$0 < \beta < 1$$

Since F is continuous, as a differentiable function, we can use the IVT and the monotonicity of F to conclude that:

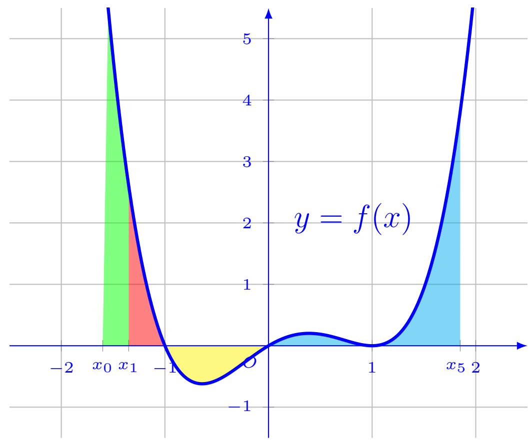
(a) There is exactly one $x_0 \in (-\infty, -1)$ such that $F(x_0) = -1$.

(b) There is exactly one $x_1 \in (-\infty, -1)$ such that $F(x_1) = 0$.

(c) $F(0) = 0$

(d) There is exactly one $x_5 \in (1, +\infty)$ such that $F(x_5) = 1$.

We can find approximate values for these x_i by looking at the graph of f:



(a) We know that x_1 is the point such that the red area is equal to the yellow one, so that $F(x_1) = 0$, we read graphically $x_1 \simeq -1.35$.

(b) We know that x_0 is the point such that the green area is equal to 1, so that $F(x_0) = -1$, we read graphically $x_0 \simeq -1.60$.

(c) We know that x_5 is the point such that the blue area is equal to 1, so that $F(x_5) = 1$, we read graphically $x_5 \simeq 1.85$.

Therefore the critical points are:
$x_0 \simeq -1.60$, $x_1 = -1.35$, $x_2 = -1$, $x_3 = 0$, $x_4 = 1$ and $x_5 \simeq 1.85$.

We still need to figure out if they are local extrema or not.

- Notice that

$$0 < x < y \Rightarrow 0 < F(x) < F(y) \Rightarrow 0 < H(x) < H(y)$$

  Hence H is strictly increasing on $(0, +\infty)$.
  So H has no local extrema at x_4 and x_5.

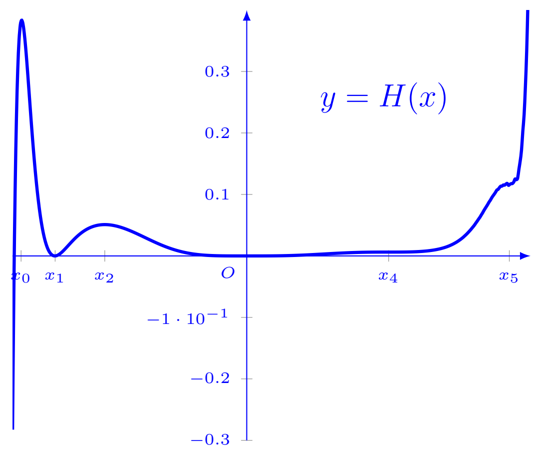
- We know that $H(x_1) = H(x_3) = 0$ whereas H is positive around these points, so they are local min.

- Since f is continuous on $[x_1, x_3]$, we know by the EVT that f has a min and a max on this interval. The max is among the endpoints and the critical points x_2. Since $H(x_1) = H(x_3) = 0$ and $H(x_2) = F(F(x_2)) = F(1) > 0$, we deduce that $H(x_2)$ is a local max.

- Since f is continuous on $[-10, x_1]$, we know by the EVT that f has a min and a max on this interval.
  The max is among the endpoints and the critical point x_0.
  Since $H(-10) < 0$, $H(x_1) = 0$ and $H(x_0) = F(F(x_0)) = F(-1) = \alpha > 0$, we deduce that $H(x_0)$ is a local max.

<u>Conclusion :</u>

- There is a local max at $x_0 \simeq -1.60$.
- There is a local min at $x_1 \simeq -1.35$.
- There is a local max at $x_2 = -1$.
- There is a local min at $x_3 = 0$.



3. In this problem, you are going to compute the exact value of the integral $I = \int_{-2}^{1} (x^2 + 1)\, dx$ using Riemann sums. Let us call $f(x) = x^2 + 1$. Since f is continuous on $[-2, 1]$, we know it is integrable. Hence, its value can be computed using Riemann sums as video 7.11 explains.

For every natural number n, let us call P_n the partition that splits $[-2, 1]$ into n *equal* sub-intervals. Notice that $\lim_{n\to\infty} ||P_n|| = 0$. Hence, we can write $I = \lim_{n\to\infty} S^{\star}_{P_n}(f)$ where $S^{\star}_{P_n}(f)$ is any Riemann sum for f and P_n. In particular, to make things simpler, we will use Riemann sums always choosing the right end-point to evaluate f on each subinterval.

(a) What is the length of each sub-interval in P_n?

The length of $[-2, 1]$ is $1 - (-2) = 3$.
Since P_n consists in breaking $[-2, 1]$ into n subintervals of same length, we obtain that the length of each subinterval is $\frac{3}{n}$.

(b) Let us write $P_n = \{x_0, x_1, \ldots, x_n\}$. Find a formula for x_i in terms of i and n.

$$x_i = -2 + i\frac{3}{n} = -2 + \frac{3i}{n}$$

(c) Since we are using the right-endpoint, it means we are picking $x_i^{\star} = x_i$. Use your above answers to obtain an expression for $S^{\star}_{P_n}(f)$ in the form of a sum with sigma notation.

$$\begin{aligned}
S^{\star}_{P_n}(f) &= \sum_{i=1}^{n} \Big((x_i - x_{i-1}) f(x_i^{\star}) \Big) \\
&= \sum_{i=1}^{n} \Big((x_i - x_{i-1}) f(x_i) \Big) \\
&= \sum_{i=1}^{n} \left(\frac{3}{n} f\left(-2 + \frac{3i}{n} \right) \right) \\
&= \frac{3}{n} \sum_{i=1}^{n} \left(\left(-2 + \frac{3i}{n} \right)^2 + 1 \right)
\end{aligned}$$

(d) Using the formulas

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \quad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$

if needed, add up the expression you got to obtain a nice, compact formula for $S^{\star}_{P_n}(f)$ without any sums or sigma symbols.

$$\begin{aligned}
S^{\star}_{P_n}(f) &= \frac{3}{n}\sum_{i=1}^{n}\left(\left(-2+\frac{3i}{n}\right)^2+1\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9}{n^2}i^2-\frac{12}{n}i+5\right) \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{15}{n}\sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{15}{n}n \\
&= \frac{27n(n+1)(2n+1) - 108n^2(n+1) + 90n^3}{6n^3} \\
&= \frac{36n^3 - 27n^2 + 27n}{6n^3} \\
&= \frac{12n^3 - 9n^2 + 9n}{2n^3}
\end{aligned}$$

(e) Calculate $\lim_{n\to\infty} S^{\star}_{P_n}(f)$. This number will be the exact value of $\int_{-2}^{1}(x^2+1)dx$.

Method 1: with the simplified form.

$$\begin{aligned}
\lim_{n\to\infty} S^{\star}_{P_n}(f) &= \lim_{n\to\infty}\frac{12n^3-9n^2+9n}{2n^3} \\
&= \frac{12}{2} \\
&= 6
\end{aligned}$$

Method 2: without the simplified form.

$$\begin{aligned}
\lim_{n\to\infty} S^{\star}_{P_n}(f) &= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + 15\right) \\
&= \frac{27\cdot 2}{6} - \frac{36}{2} + 15 \\
&= 9 - 18 + 15 \\
&= 6
\end{aligned}$$

Hence

$$\int_{-2}^{1} (x^2+1)dx = 6$$

(f) (**Do not submit.**) Now repeat all the previous steps using left endpoints instead of right endpoints. You should get the exact same final answer.

$$\begin{aligned}
S^{\star}_{P_n}(f) &= \sum_{i=1}^{n} \Big((x_i - x_{i-1}) f(x_i^{\star}) \Big) \\
&= \sum_{i=1}^{n} \Big((x_i - x_{i-1}) f(x_{i-1}) \Big) \\
&= \sum_{i=1}^{n} \left(\frac{3}{n} f\left(-2 + \frac{3(i-1)}{n} \right) \right) \\
&= \frac{3}{n} \sum_{i=1}^{n} \left(\left(-2 + \frac{3(i-1)}{n} \right)^2 + 1 \right) \\
&= \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{18i}{n^2} - \frac{12i}{n} + \frac{9}{n^2} + \frac{12}{n} + 5 \right) \\
&= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{54}{n^3} \cdot \frac{n(n+1)}{2} - \frac{36}{n^2} \cdot \frac{n(n+1)}{2} + \frac{27}{n^3} n + \frac{36}{n^2} n + \frac{15}{n} \cdot n
\end{aligned}$$

Hence

$$\lim_{n\to\infty} S^{\star}_{P_n}(f) = \frac{27 \cdot 2}{6} - 0 - \frac{36}{2} + 0 + 0 + 15 = 6$$

(g) (**Do not submit.**) Verify that your answer is correct using antiderivatives and FTC 2.

$$\int_{-2}^{1} (x^2+1)dx = \left[\frac{x^3}{3} + x \right]_{-2}^{1} = \frac{1}{3} + 1 + \frac{8}{3} + 2 = 6$$