MAT 137Y: Calculus!
Problem Set 7
Due on Thursday, January 31 by 11:59pm via crowdmark

Instructions:

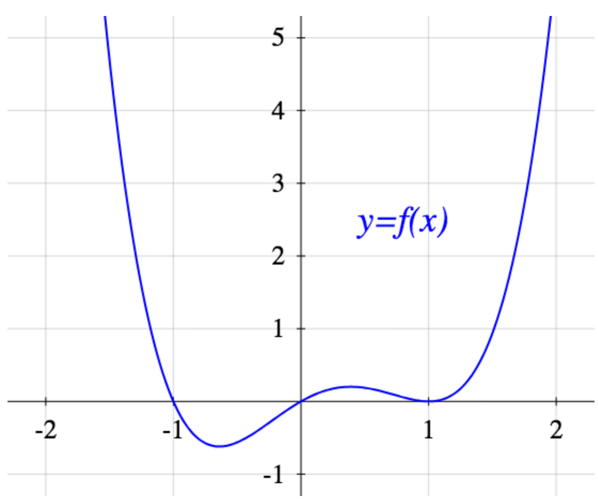
- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about integrals, Riemann sums and the Fundamental Theorem of Calculus (Playlist 7 and 8).

1. Let $a, b, c, k \in \mathbb{R}$. Compute the following limit

$$\lim_{x \to 0} \frac{\int_{ax}^{bx} \left[\int_{ct}^{kt} e^{-s^2} ds \right] dt}{\cos x - 1}$$

Make sure you explain what you are doing and to justify the steps you take. You will not get any credit for a bunch of calculations without any words. If you find the calculation hard, it may be helpful to give names to some of the intermediate functions.

2. Below is the graph of the function f:



The domain of f is $\mathbb{R}$ and the graph continues to the right and to the left as you expect. We define a new function H by

$$H(x) = \int_0^{\int_0^x f(t)dt} f(s)ds$$

How many local maxima and local minima does H have? Give the approximate x-coordinate for each one of them.

Hint: If you are having trouble computing the derivative of H, we recommend again that you give names to the intermediate functions.

3. In this problem, you are going to compute the exact value of the integral $I = \int_{-2}^{1} \left(x^2+1\right) dx$ using Riemann sums. Let us call $f(x) = x^2 + 1$. Since f is continuous on $[-2, 1]$, we know it is integrable. Hence, its value can be computed using Riemann sums as video 7.11 explains.

   For every natural number n, let us call P_n the partition that splits $[-2, 1]$ into n *equal* sub-intervals. Notice that $\lim_{n\to\infty} ||P_n|| = 0$. Hence, we can write $I = \lim_{n\to\infty} S^\star_{P_n}(f)$ where $S^\star_{P_n}(f)$ is any Riemann sum for f and P_n. In particular, to make things simpler, we will use Riemann sums always choosing the right end-point to evaluate f on each subinterval.

   (a) What is the length of each sub-interval in P_n?

   (b) Let us write $P_n = \{x_0, x_1, \ldots, x_n\}$. Find a formula for x_i in terms of i and n.

   (c) Since we are using the right-endpoint, it means we are picking $x_i^\star = x_i$. Use your above answers to obtain an expression for $S^\star_{P_n}(f)$ in the form of a sum with sigma notation.

   (d) Using the formulas

   $$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \quad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$

   if needed, add up the expression you got to obtain a nice, compact formula for $S^\star_{P_n}(f)$ without any sums or sigma symbols.

   (e) Calculate $\lim_{n\to\infty} S^\star_{P_n}(f)$. This number will be the exact value of $\int_{-2}^{1}(x^2+1)dx$.

   (f) (**Do not submit.**) Now repeat all the previous steps using left endpoints instead of right endpoints. You should get the exact same final answer.

   (g) (**Do not submit.**) Verify that your answer is correct using antiderivatives and FTC 2.