MAT 137Y: Calculus!
Problem Set 6
Sample solutions

1. Let f be a function with domain $\mathbb{R}$. Assume f is decreasing and bounded below. Let A be the infimum of f. Prove that

$$\lim_{x\to\infty} f(x) = A$$

<u>METHOD 1:</u> (using the ε-characterization of the infimum)

We know that:

(a) f is decreasing, i.e.

$$\forall x, y \in \mathbb{R},\ x < y \Rightarrow f(x) > f(y) \tag{1}$$

(b) $A = \inf\{f(x) \ :\ x \in \mathbb{R}\}$, i.e.

$$\begin{cases} \forall x \in \mathbb{R},\ A \leq f(x) & (2) \\ \forall \varepsilon > 0,\ \exists x_0 \in \mathbb{R},\ f(x_0) < A + \varepsilon & (3) \end{cases}$$

We want to show that $\lim_{x\to+\infty} f(x) = A$, i.e.

$$\forall \varepsilon > 0,\ \exists M \in \mathbb{R},\ \forall x \in \mathbb{R},\ \big(x > M \Rightarrow |f(x) - A| < \varepsilon\big)$$

- Let us fix an arbitrary $\varepsilon > 0$.
- I use this same value of ε in (3).
  Then there exists $x_0 \in \mathbb{R}$ such that $f(x_0) < A + \varepsilon$.
  I take $M = x_0$.
- Let $x \in \mathbb{R}$. Assume that $x > M$. I want to prove that $|f(x) - A| < \varepsilon$.
  - By (1), since $x > M$, we have that $f(x) < f(M) < A + \varepsilon$.
  - By (2), $f(x) \geq A > A - \varepsilon$.

  I have proven that $A - \varepsilon < f(x) < A + \varepsilon$. This is equivalent to $|f(x) - A| < \varepsilon$.

Q.E.D.

METHOD 2: (without the ε-characterization of the infimum)

We know that:

(a) f is decreasing, i.e.

$$\forall x, y \in \mathbb{R},\ x < y \Rightarrow f(x) > f(y) \tag{4}$$

(b) $A = \inf\{f(x)\ :\ x \in \mathbb{R}\}$, which means that

- A is a lower bound of f, i.e.

$$\forall x \in \mathbb{R},\ A \leq f(x) \tag{5}$$

- and, it is the greatest one, i.e.

$$\text{if } B \text{ is a lower bound of } f \text{ then } A \geq B \tag{6}$$

Actually, we are going to use the contrapositive of (6):

$$\text{If } A < B \text{ then } B \text{ isn't a lower bound of } f \tag{6'}$$

We want to show that $\lim_{x\to+\infty} f(x) = A$, i.e.

$$\forall \varepsilon > 0,\ \exists M \in \mathbb{R},\ \forall x \in \mathbb{R},\ \big(x > M \Rightarrow |f(x) - A| < \varepsilon\big)$$

- Let us fix an arbitrary $\varepsilon > 0$.
- Since $A + \varepsilon > A$, we derive from (6') that $A + \varepsilon$ isn't a lower bound of f. This means there exists $M \in \mathbb{R}$ such that $f(M) < A + \varepsilon$.
- Let $x \in \mathbb{R}$. Assume that $x > M$. I will show that $|f(x) - A| < \varepsilon$.
  - By (4), since $x > M$, we have $f(x) < f(M) < A + \varepsilon$.
  - By (5), $f(x) \geq A > A - \varepsilon$.

  I have proven that $A - \varepsilon < f(x) < A + \varepsilon$. This is equivalent to $|f(x) - A| < \varepsilon$.

Q.E.D.

2. Consider the set

$$B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$
$$= \left\{x \in \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ s.t. } n > 0 \text{ and } x = \frac{1}{n}\right\}$$

I define the function g by the equation

$$g(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \notin B \end{cases}$$

In this question you are going to study the integrability of the function g on $[0, 1]$.

(a) What is the upper integral $\overline{I_0^1}(g)$?

For every partition, the supremum of g on every subinterval is always 1.
Therefore every upper sum is 1.
Therefore the upper integral is $\overline{I_0^1}(g) = \sup\{1\} = 1$.

We can also write a more formal proof. Let $P = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ be a partition of $[0, 1]$. Then the P-upper sum of g is

$$U_P(g) = \sum_{k=1}^{N} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} g \right)$$

For each k, we know there exists $a \in [x_{k-1}, x_k]$ which is not a rational number.
In particular, $a \notin B$ and $g(a) = 1$.
Moreover, $\forall x \in [x_{k-1}, x_k]$, $g(x) \leq 1$.
Hence $\sup_{[x_{k-1}, x_k]} g = 1$.
Therefore

$$U_P(g) = \sum_{k=1}^{N} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} g \right) = \sum_{k=1}^{N} (x_k - x_{k-1}) = x_N - x_0 = 1 - 0 = 1$$

Thus $U_P(g) = 1$ for any partition P of $[0, 1]$.
Finally, $\overline{I_0^1}(g) = \inf\{ \text{ upper sums } \} = \inf\{1\} = 1$.

Q.E.D.

(b) Prove the following claim:

"For every positive integer n, and for every $\varepsilon > 0$,
there exists a partition P of $[0,1]$ such that $L_P(g) > 1 - \frac{1}{n} - \varepsilon$."

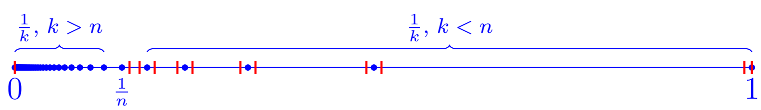
<u>Preliminary work: Introducing a family of partitions</u>

I will introduce a family of partitions that have large lower sums. Given $n \in \mathbb{N}$ such that $n \geq 2$, and given $\alpha > 0$ small enough (I will explain what this means in a moment), I define the partition

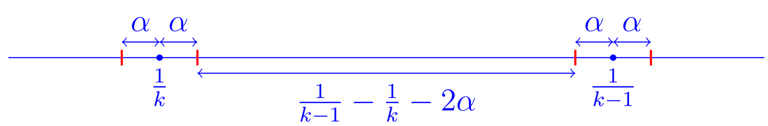
$$P_{n,\alpha} = \left\{0 < \frac{1}{n} + \alpha < \frac{1}{n-1} - \alpha < \frac{1}{n-1} + \alpha < \cdots < \frac{1}{k} - \alpha < \frac{1}{k} + \alpha < \cdots < \frac{1}{2} - \alpha < \frac{1}{2} + \alpha < 1 - \alpha < 1\right\} \tag{7}$$

This partition contains all the points in $\left(\frac{1}{n}, 1\right)$ which are at distance α of one point in B.

The two following figures may help you to vizualize this partition.
First this is what $P_{n,\alpha}$ looks like (the boundaries of the subintervals of the partition are in red, the points in B are marked by $\bullet$):



Next, this is a zoom around two consecutive elements of $B \cap [0,1]$ where $k \leq n$:



$\Big($*The function g vanishes on the subintervals containing an element of B, represented by a $\bullet$ on the figures. Then, the idea consists in choosing α small enough so that the subintervals without an element of B are big enough: the bigger they are, the bigger the lower sum is.*$\Big)$

Lemma 1: This partition is well-defined as long as $\alpha < \dfrac{1}{2n(n-1)}$.

Proof of Lemma 1: For this partition to be well-defined, the inequalities in (7) need to be satisfied. More specifically, we need that

$$\frac{1}{k} + \alpha < \frac{1}{k-1} - \alpha$$

for $1 < k \leq n$. This condition is equivalent to

$$2\alpha < \frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)},$$

which will be true for all $1 < k \leq n$ as log as

$$2\alpha < \frac{1}{n(n-1)}$$

□

Lemma 2: $L_{P_{n,\alpha}}(g) = 1 - \dfrac{1}{n} - 2\alpha(n-1)$

Proof of Lemma 2:

- If a subinterval of the partition contains an element in B, then the infimum of g on it is 0, and that subinterval does not contribute to the lower sum.
- If a subinterval does not contain any element in B, then g is constantly equal to 1 on that subintervals.

Thus, the lower sum is the sum of the lengths of the subintervals that do not intersect B. Equivalently, the lower sum is 1 minus the sum of the lengths of the subintervals that do intersect B. These include the first subinterval (with length $1/n$) and $2(n-1)$ other subintervals (each with length α). Therefore:

$$L_{P_{n,\alpha}}(g) = 1 - \left[\frac{1}{n} + 2(n-1)\cdot\alpha\right].$$

□

Proof of the original claim

- Let n be a positive integer. Let $\varepsilon > 0$.
- If $n = 1$, then the partition $P = \{0 < 1\}$ will work. For the rest of this proof we will assume that $n > 1$.
- Let $\alpha = \frac{1}{2} \min\left(\frac{\varepsilon}{2(n-1)}, \frac{1}{2n(n-1)}\right)$. Then $\alpha > 0$.
- I take the partition $P_{n,\alpha}$.

  By Lemma 1, this partition is well-defined because $\alpha < \frac{1}{2n(n-1)}$.
- By Lemma 2, the lower sum of this partition satisfies

$$L_{P_{n,\alpha}} = 1 - \frac{1}{n} - 2\alpha(n-1) > 1 - \frac{1}{n} - \frac{\varepsilon}{2} > 1 - \frac{1}{n} - \varepsilon$$

Q.E.D.

(c) Prove the following claim:

"*For every* $\varepsilon > 0$, *there exists a partition* P *of* $[0,1]$ *such that* $L_P(g) > 1 - \varepsilon$."

Let $\varepsilon > 0$.

I choose a positive integer n large enough such that $\frac{1}{n} < \frac{\varepsilon}{2}$. For example, I can take $n = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$.

Then I use the result from Question 2b for the values n and $\frac{\varepsilon}{2}$. This produces a partition P such that

$$L_P(g) > 1 - \frac{1}{n} - \frac{\varepsilon}{2} > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon$$

Q.E.D.

(d) What is the lower integral $\underline{I_0^1}(g)$?

We are going to prove that $\underline{I_0^1}(g) = 1$.
Recall that $\underline{I_0^1}(g) = \sup \{L_P(g) \;:\; P \text{ is a partition of } [0,1]\}$.

METHOD 1:

Notice that in Questions 2b and 2c we did not compute *all* partitions, so it is not okay to simply take the supremum of the lower sums that we have computed so far.

I will prove that $\underline{I_0^1}(g) \leq 1$ and that $\underline{I_0^1}(g) \geq 1$ separately.

- For any partition P of $[0,1]$, Question 2a tell us that $L_P(g) \leq U_P(g) = 1$. Thus 1 is an upper bound of the set of lower sums. By definition of "lowest upper bound", we conclude that $\underline{I_0^1}(g) \leq 1$.
- On Question 2c we have proven that for every $\varepsilon > 0$, there exists a partition P such that $1 - \varepsilon < L_P(g)$. Since $L_P(g) \leq \underline{I_0^1}(g)$ this guarantees that

$$\forall \varepsilon > 0, \quad 1 - \varepsilon < \underline{I_0^1}(g).$$

Therefore, $1 \leq \underline{I_0^1}(g)$.

Q.E.D.

METHOD 2:

According to the ε-characterization of supremum, to prove that $\underline{I_0^1}(g) = 1$, we need to show two things:

- For every partition P of $[0,1]$, $L_P(g) \leq 1$
- For every $\varepsilon > 0$, there exists a partition P of $[0,1]$ such that $1 - \varepsilon < L_P(g)$

But I have already proven both!

- The first condition is true because, from Question 2a, every partition P satisfies $U_P(g) = 1$ and also $L_P(g) \leq U_P(g)$.
- The second condition is exactly what we proved in Question 2c.

Q.E.D.

(e) Is g integrable on $[0,1]$?

According to Questions 2a and 2d, we have

$$\underline{I_0^1}(g) = \overline{I_0^1}(g) = 1.$$

Since they are equal, by definition g is integrable on $[0,1]$ and

$$\int_0^1 g(x)dx = 1$$

Q.E.D.