MAT 137Y: Calculus!
Problem Set 6
Due on MONDAY, January 21 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about suprema, infima, and the definition of integral (Videos 7.1–7.9).

1. Let f be a function with domain $\mathbb{R}$. Assume f is decreasing and bounded below. Let A be the infimum of f. Prove that

$$\lim_{x\to\infty} f(x) = A$$

Notes: You will need to use the definition of infimum of a function and the definition of $\lim_{x\to\infty} f(x)$. Do not make any unwarranted assumptions about the function f; for example, do not assume that f is continuous or that $\lim_{x\to\infty} f(x)$ exists.

2. Consider the set

$$\begin{aligned} B &= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} \\ &= \left\{x \in \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ s.t. } n > 0 \text{ and } x = \frac{1}{n}\right\} \end{aligned}$$

I define the function g by the equation

$$g(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \notin B \end{cases}$$

In this question you are going to study the integrability of the function g on $[0, 1]$. Before you begin, sketch the graph of this function to understand it better. As always, make sure to justify all your answers.

(a) What is the upper integral $\overline{I_0^1}(g)$?

(b) Prove the following claim:

"*For every positive integer n, and for every $\varepsilon > 0$, there exists a partition P of $[0,1]$ such that $L_P(g) > 1 - \frac{1}{n} - \varepsilon$.*"

Hint: Before working on this question, answer the following extra questions in order. **You do not need to submit these extra questions**, but thinking about them will make answering 2b easier.

- Construct one partition P of $[0,1]$ such that $L_P(g) = 0$
- Construct one partition P of $[0,1]$ with exactly 4 points (3 subintervals) such that $L_P(g) = 0.48$.
- Construct one partition P of $[0,1]$ with exactly 4 points (3 subintervals) such that $L_P(g) = 0.499$.
- Construct one partition P of $[0,1]$ with exactly 6 points (5 subintervals) such that $L_P(g) = \frac{2}{3} - 0.001$.

(c) Prove the following claim:

"*For every $\varepsilon > 0$, there exists a partition P of $[0,1]$ such that $L_P(g) > 1 - \varepsilon$.*"

Hint: Use your answer to Question 2b. Mind your quantifiers and your proof structure.

(d) What is the lower integral $\underline{I_0^1}(g)$?

(e) **[Do not submit]** Is g integrable on $[0,1]$?