MAT 137Y: Calculus!
Problem Set 5
Sample solutions

1. Alfonso is relaxing in the center of a square pool when suddenly he hears a yell. Ivan is standing at the corner of the pool, looking angry. Alfonso chooses a direction and starts swimming towards the side of the pool. Even though he does not know Ivan's exact speed, Alfonso knows he can outrun Ivan, so if he exits the pool before Ivan gets there, he is safe. Unfortunately, he is a very slow swimmer, and Ivan has started running around the edges of the pool towards Alfonso's exit point. Ivan is afraid of water and won't enter the pool. Also, once Alfonso chooses a direction, he never turns and he always remains straight, no matter how hard it may be.

   At what point should Alfonso try to exit the pool?

   The first step in this problem is figure out what we need to optimize. Let S_A, T_A, and V_A be the distance Alfonso has to swim, the time it takes him, and his speed, respectively. Let S_I, T_I, and V_I be the distance Ivan has to run, the time it takes him, and his speed, respectively. Alfonso would like to have $T_A < T_I$ in order to escape. Since

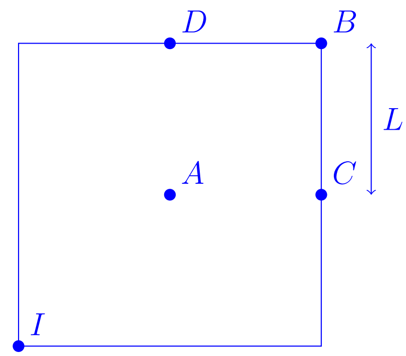
   $$T_A = \frac{S_A}{V_A} \quad \text{and} \quad T_I = \frac{S_I}{V_I},$$

   the inequality $T_A < T_I$ becomes $\dfrac{S_A}{V_A} < \dfrac{S_I}{V_I}$ or, equivalently,

   $$\frac{V_I}{V_A} < \frac{S_I}{S_A}. \tag{1}$$

   Unfortunately, Alfonso does not know Ivan's running speed precisely. In order to maximize his likelihood of satisfying (1), Alfonso should try to make the quantity $\dfrac{S_I}{S_A}$ as large as possible. That will allow him to escape for the largest possible range of values of V_I. This may not be enough (if V_I is much larger than V_A, then it does not matter where Alfonso tries to exit; he will always fail) but it is the best he can do. Thus, we will look for the maximum value of $\frac{S_I}{S_A}$. Equivalently, we could also look for the minimum value of $\dfrac{S_A}{S_I}$.

   Next, we can simplify the problem a little bit. Let L be half the length of the side of the pool (i.e., so the pool has sides of length $2L$). Let A be Alfonso's initial position. Let I be the corner where Ivan starts, and let B be the corner of the pool opposite I. Let C and D be the two points, midway through a side of the pool, closest to B:



Alfonso should only try to exit at points between B and C or between B and D. For any other point in the boundary of the pool, there is a point between B and C such that the distance Alfonso has to swim is smaller but the distance Ivan has to run is larger, and so would always be preferable. Thus, we can reduce our domain to these two line segments. Moreover, due to the symmetry of the pool, the segment from B to C is equivalent to the segment from B to D. We will do our calculations for the segment from B to C only.

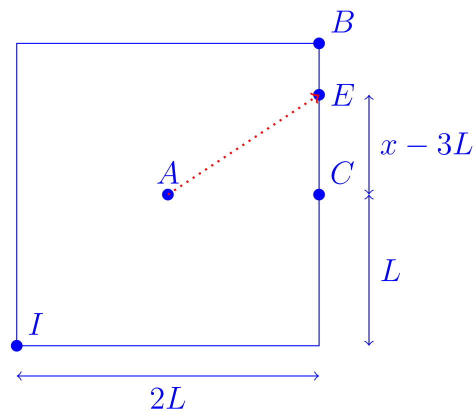
In conclusion, we want to find the maximum of $\frac{S_I}{S_A}$ for points on the segment between B and C.

Next, we need to choose a variable to describe the possible exit points between B and C. There are at many choices we can make; some are distances, some are angles. We will solve the problem with one of each.

METHOD 1:

- Modelling

  Let E be the exit point Alfonso chooses. Let $x = S_I$ be the distance Ivan has to run. We will use x as the variable in this method.



  Since E lies between B and C, we get that $x \in [3L, 4L]$. To calculate the distance S_A that Alfonso has to swim, we look at the right triangle AEC. Using the Pythagorean Theorem, we have

$$S_A^2 \ = \ (AE)^2 \ = \ (AC)^2 + (CE)^2 \ = \ L^2 + (x - 3L)^2 \ = \ 10L^2 - 6Lx + x^2.$$

  Now we want to minimize the quantity $\frac{S_A}{S_I}$. Equivalently, since the squaring function is increasing on $[0, \infty)$, we want to minimize the function

$$f(x) = \frac{S_A^2}{S_I^2} = \frac{10L^2 - 6Lx + x^2}{x^2}.$$

  In summation, we have reduced the problem to finding the maximum of the function

$$f(x) = 1 - 6\frac{L}{x} + 10\frac{L^2}{x^2}, \qquad 3L \leq x \leq 4L.$$

- Calculus

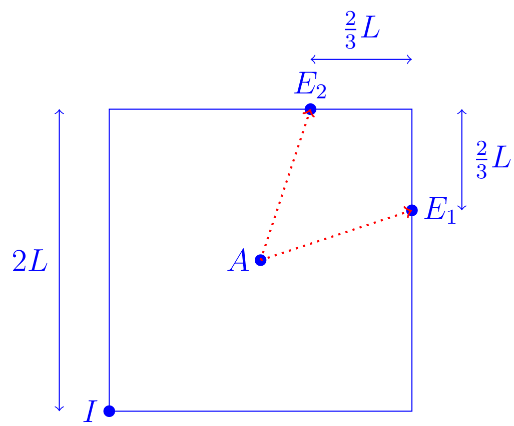
  By the EVT, f must have a minimum, which we know must be at a critical point or at an endpoint of the domain. f is differentiable on $[3L, 4L]$ and $f'(x) = \dfrac{2L}{x^3}(3x - 10L)$. The only critical point is $x = \frac{10}{3}L$.

x	$3L$		$\frac{10}{3}L$		$4L$
$f'(x)$		$-$	0	$+$	
$f(x)$	$\frac{1}{9}$	$\searrow$	$\frac{1}{10}$	$\nearrow$	$\frac{1}{8}$

So the minimum of f is reached at $x = \frac{10}{3}L$.

- Interpretation

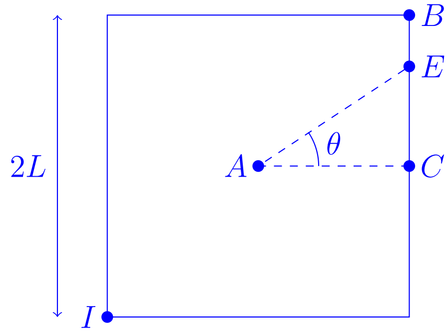
  Alfonso should exit the pool at a distance of $4L - \frac{10}{3}L = \frac{2}{3}L$ from the corner opposite the one where Ivan is initially located.



METHOD 2:

- Modelling

  Let E be the exit point Alfonso chooses. Let θ be the angle from AC to AE. We will use θ as the variable in this method.



  We want to maximize the function $g(\theta) = \dfrac{S_I}{S_A}$, so we need to write it in terms of θ.

  – Since E must be between B and C, we get the domain $\theta \in [0, \pi/4]$.
  – We know that $AC = L$.
  – We also know that $\cos\theta = \frac{AC}{AE}$.
    Therefore, the distance Alfonso must swim is $S_A = AE = L\sec\theta$.
  – In addition, we have $\tan\theta = \frac{CE}{AC}$.
    Therefore, the distance Ivan must run is $S_I = 3L + CE = 3L + L\tan\theta$.
  – Putting it all together, we have

$$g(\theta) \;=\; \frac{S_I}{S_A} \;=\; \frac{3L + L\tan\theta}{L\sec\theta} \;=\; 3\cos\theta + \sin\theta.$$

  In summation, we have reduced the problem to finding the maximum of the function

$$g(\theta) = 3\cos\theta + \sin\theta, \qquad 0 \le \theta \le \frac{\pi}{4}.$$

- Calculus

  By the EVT, g must have a maximum, which we know must be at a critical point or at an endpoint of the domain. g is differentiable everywhere and

$$g'(\theta) = -3\sin\theta + \cos\theta.$$

  Thus, $g'(\theta) = 0$ if and only if $\tan\theta = \frac{1}{3}$. This produces a single critical point in our domain, which we will call θ_0:

$$\theta_0 = \arctan\frac{1}{3}.$$

Notice that

$$\tan\theta_0 = \frac{1}{3}, \quad \sin\theta_0 = \frac{1}{\sqrt{10}}, \quad \text{and} \quad \cos\theta_0 = \frac{3}{\sqrt{10}}.$$

The maximum of g must occur at θ_0, or at one of the endpoints of the domain. We compute:

$$\begin{aligned} g(0) &= 3\cos 0 + \sin 0 = 3 \\ g(\theta_0) &= 3\cos\theta_0 + \sin\theta_0 = 3\left(\frac{3}{\sqrt{10}}\right) + \left(\frac{1}{\sqrt{10}}\right) = \sqrt{10} \\ g(\tfrac{\pi}{4}) &= 3\cos(\tfrac{\pi}{4}) + \sin(\tfrac{\pi}{4}) = 3\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) = \sqrt{8} \end{aligned}$$

The largest of these three numbers is $\sqrt{10}$. Hence, g reaches its maximum at θ_0.

- Interpretation

  Notice that at θ_0 we have $CE = L\tan\theta_0 = \frac{1}{3}L$. Hence, $BE = L - \frac{1}{3}L = \frac{2}{3}L$. We get the exact same interpretation that we got with Method 1.

2. Consider the following FALSE theorem and BAD proof.

False theorem
Let h be a function defined on an open interval I. Assume h is differentiable on I. Then h' is continuous on I.

Bad proof
Let $a \in I$. By definition, $h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$.
Since h is continuous, the limit of the numerator is 0. The limit of the denominator is also 0. Since h is differentiable, I can apply L'Hôpital's Rule.

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \overset{L'H}{=\!=} \lim_{x \to a} \frac{h'(x) - 0}{1 - 0} = \lim_{x \to a} h'(x).$$

I have proven that $h'(a) = \lim_{x \to a} h'(x)$. By definition, h' is continuous. □

(a) Explain the error in the proof.

Recall the statement of l'Hôpital's rule:
IF

i. $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$,

ii. f and g are differentiable near a (except maybe at a),

iii. g and g' do not vanish near a (except maybe at a), and

iv. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or is $+\infty$ or $-\infty$.

THEN
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

In the situation of the question, we do not know whether the assumption (iv) is satisfied, so we cannot apply l'Hôpital's rule.

Counterexample: For example, the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is differentiable on $\mathbb{R}$, but $\lim_{x \to 0} f'(x)$ does not exist and so we cannot apply l'Hôpital's rule as in the **Bad Proof** for $a = 0$ (so the proof does not work for this function).

Moreover, since f' is not continuous at 0, not only does the proof not work but the theorem is also false.

Interesting side fact: If a function f is differentiable on an interval I then its derivative f' satisfies the conclusion of the IVT even if f' is not continuous. This is called *Darboux's theorem.*

(b) "Fix" the theorem. (In other words, modify the statement of the theorem a little bit, either changing the hypotheses or the conclusion, so that it is true. There may be more than one way to do it.) You do not need to write the proof.

Here are some true thorems.

- Let h be a function defined on an open interval I. Assume h is differentiable on I **and that for all** $a \in I$**,** $\lim_{x \to a} h'(x)$ **exists**. Then h' is continuous on I.
- Let h be a function defined on an open interval I. Assume h is differentiable on I. Let $a \in I$.
  IF $\lim_{x \to a} h'(x)$ exists, THEN $\lim_{x \to a} h'(x) = h'(a)$.
- Let h be a function defined on an open interval I. Assume h is differentiable on I. Then h' does not have any removable discontinuity on I, it does not have any jump discontinuity on I, and it does not have any vertical asymptote on I.

The following are also true statements, but they are not honest ways to "fix" the bad theorem. They are just repetitions of an old theorem we already knew.

- Let h be a function defined on an open interval I. Assume h is differentiable on I. Then h is continuous on I.
- Let h be a function defined on an open interval I. Assume h is twice differentiable on I. Then h' is continuous on I.

3. Let I be an open interval. Let $a \in I$. Let f be a function defined on I. Assume that f is continuous at a and that f is differentiable near a (except possibly at a). Consider the following two definitions:

- f *has a vertical tangent line at* a when $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \infty$ or $-\infty$.
- f *is funky at* a when $\lim_{x \to a} f'(x) = \infty$ or $-\infty$.

Is each one of the following statements true or false? If true, prove it. If false, construct a counterexample

(a) IF f is funky at a, THEN f has a vertical tangent line at a.

The above statement is TRUE. Here is a proof:

- Assume that f is funky.
- Then
$$\lim_{x \to a} \frac{f'(x) - 0}{1 - 0} = \lim_{x \to a} f'(x) = \infty \text{ or } -\infty.$$
Hence the argument from the "bad proof" in Question 2 is valid in this case and we can apply l'Hôpital's rule:
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \overset{L'H}{=\!=} \lim_{x \to a} \frac{f'(x) - 0}{1 - 0} = \lim_{x \to a} f'(x) = \infty \text{ or } -\infty$$
- Hence f has a vertical tangent line at a.

□

(b) IF f has a vertical tangent line at a, THEN f is funky at a.

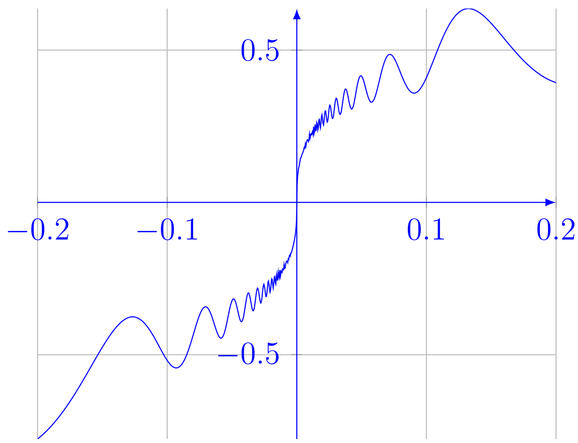
The above statement is FALSE. Here is a counterexample.

Let

$$f(x) = \begin{cases} \sqrt[3]{x} + x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

defined on $\mathbb{R} = (-\infty, \infty)$.

The graph of this function looks like:



- First, we will show that f has a vertical tangent line. We will show that $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \infty$.

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \left[\frac{1}{x^{\frac{2}{3}}} + \sin\left(\frac{1}{x}\right) \right].$$

The first term in the brackets has limit ∞ while the second is bounded between -1 and 1, so the limit of the sum is ∞. (If you prefer to be more formal, use the version of the Squeeze Theorem for infinite limits.)

- Second, we will show that f is not funky at 0. I will show that $\lim_{x\to 0} f'(x) \neq \infty$ $-\infty$.

For every $x \in \mathbb{R} \setminus \{0\}$, f is differentiable at x and

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}} + \sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x} = \sin\left(\frac{1}{x}\right) + \frac{\frac{1}{3}x^{1/3} - \cos\left(\frac{1}{x}\right)}{x}.$$

We will first give an intuitive argument, and then a more formal one.

Here is a "handwavy" argument. We have written $f'(x)$ as the sum of the two terms.

- The first term, $\sin\frac{1}{x}$, is bounded between -1 and 1, and is going to become irrelevant since the other term is much bigger in absolute value as $x \to 0$.
- As for the second term...
  * The numerator is the sum of $\frac{1}{3}x^{1/3}$, which approaches 0, and $\cos\left(\frac{1}{x}\right)$, which oscillates between -1 and 1 more and more quickly as x approaches 0. Thus, their sum also oscillates between "close to -1" and "close to 1" more and more quickly as x approaches 0.
  * The denominator approaches 0.

  Thus the second term oscillates more and more quickly between very large numbers and very small numbers as $x \to 0$. We could say that the second term "oscillates between $-\infty$ and ∞".

We can also write something more formal, using the definition of infinite limits. I will use an idea similar to that in Video 2.9, looking at two sequences of points that approach 0 where the derivative behaves very differently.
For each positive integer k, let $x_k = \frac{1}{k\pi}$. Then $\sin\frac{1}{x_k} = 0$ and $\cos\frac{1}{x_k} = \pm 1$, depending on whether k is even or odd. Thus

$$f'(x_k) = \left(\frac{1}{3}{x_k}^{1/3} \mp 1\right) k\pi$$

If we take only integers k, as $k \to \infty$, $x_k \to 0$. Then $f'(x_k)$ is negative or positive (depending on whether k is even or odd) and unbounded in absolute value.
In other words, for every $\delta > 0$, the interval $(-\delta, \delta)$ contains non-zero numbers x_k such that $f'(x_k)$ is as large as we want in absolute value and positive (when k odd) or negative (when k is even). This is enough to conclude that $\lim_{x\to 0^+} f'(x) \neq \infty$ and $\lim_{x\to 0^+} f'(x) \neq -\infty$ from the definition.

□

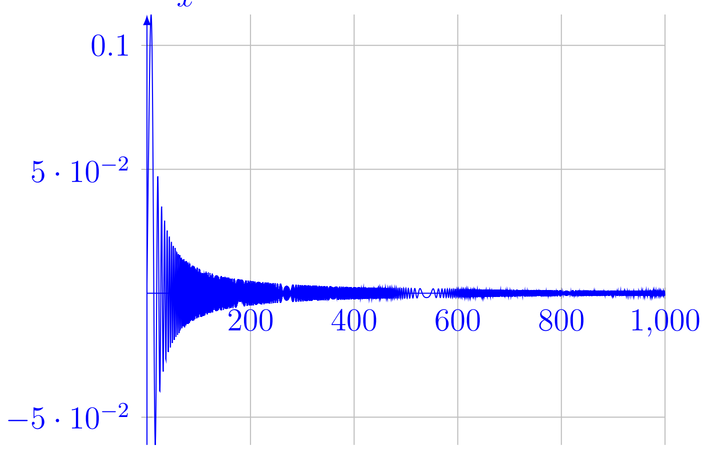
4. Let f be a function defined, at least, on an interval of the form (c, ∞) for some $c \in \mathbb{R}$. Assume f is differentiable.

Below are two claims. Are they true or false? If true, prove it. If false, provide a counterexample (and, as usual, show that your counterexample works).

(a) IF f has a horizontal asymptote as $x \to \infty$, THEN $\lim_{x\to\infty} f'(x) = 0$.

The above statement is FALSE. Here is a counterexample:

- The function $f(x) = \dfrac{\sin(x^2)}{x}$ is defined and differentiable on $(0, \infty)$.



- f admits the horizontal asymptote $y = 0$ as $x \to \infty$ since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left[\sin(x^2) \cdot \frac{1}{x}\right] = 0.$$

This can be proven either by using the Squeeze Theorem, or by noticing that f is the product of a bounded function and a function whose limit is 0.

- However $\lim_{x\to+\infty} f'(x)$ does not exist (and so in particular, $\lim_{x\to\infty} f'(x) \neq 0$).

To prove this, notice that $f'(x) = 2\cos(x^2) - \dfrac{\sin(x^2)}{x^2}$. We know that

$$\lim_{x\to\infty} \frac{\sin(x^2)}{x^2} = 0 \quad \text{(by the same argument as above)}$$

and

$$\lim_{x\to\infty} 2\cos(x^2) \text{ does not exist.}$$

If $\lim_{x\to\infty} f'(x)$ also existed, then we would have that

$$\lim_{x\to\infty} 2\cos(x^2) \ = \ \lim_{x\to\infty} \left[f'(x) + \frac{\sin(x^2)}{x^2}\right]$$

also exists by the limit law for sums, which is a contradiction.

(b) IF $\lim_{x\to\infty} f'(x) = 0$, THEN f has a horizontal asymptote as $x \to \infty$.

The above statement is FALSE. Here is a counterexample:

- The function $f(x) = \ln(x)$ is defined and differentiable on $(0, \infty)$.
- We have
$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{1}{x} = 0.$$
- But
$$\lim_{x\to\infty} f(x) = \infty$$
so f does not admit a horizontal asymptote as $x \to \infty$.