MAT 137Y: Calculus!
Problem Set 5
Due on Thursday, January 10 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about Playlist 6.

1. Alfonso is relaxing in the center of a square pool when suddenly he hears a yell. Ivan is standing at the corner of the pool, looking angry. Alfonso chooses a direction and starts swimming towards the side of the pool. Even though he does not know Ivan's exact speed, Alfonso knows he can outrun Ivan, so if he exits the pool before Ivan gets there, he is safe. Unfortunately, he is a very slow swimmer, and Ivan has started running around the edges of the pool towards Alfonso's exit point. Ivan is afraid of water and won't enter the pool. Also, once Alfonso chooses a direction, he never turns and he always remains straight, no matter how hard it may be.

   At what point should Alfonso try to exit the pool?

Before you attempt the rest of the problems, you may find it useful to first study the "standard" example of a function with a discontinuous derivative. You may have seen this function in class already, but here is a refresher: `https://tinyurl.com/mat137ps5`

2. Consider the following FALSE theorem and BAD proof.

   False theorem
   Let h be a function defined on an open interval I. Assume h is differentiable on I. Then h' is continuous on I.

Bad proof

Let $a \in I$. By definition, $h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$.
Since h is continuous, the limit of the numerator is 0. The limit of the denominator is also 0. Since h is differentiable, I can apply L'Hôpital's Rule.

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \overset{L'H}{=\!=} \lim_{x \to a} \frac{h'(x) - 0}{1 - 0} = \lim_{x \to a} h'(x).$$

I have proven that $h'(a) = \lim_{x \to a} h'(x)$. By definition, h' is continuous. $\square$

(a) Explain the error in the proof.

(b) "Fix" the theorem. (In other words, modify the statement of the theorem a little bit, either changing the hypotheses or the conclusion, so that it is true. There may be more than one way to do it.) You do not need to write the proof.

3. Let I be an open interval. Let $a \in I$. Let f be a function defined on I. Assume that f is continuous at a and that f is differentiable near a (except possibly at a). Consider the following two definitions:

   - *f has a vertical tangent line at a* when $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \infty$ or $-\infty$.
   - *f is funky at a* when $\lim_{x \to a} f'(x) = \infty$ or $-\infty$.

   Is each one of the following statements true or false? If true, prove it. If false, construct a counterexample

   (a) IF f is funky at a, THEN f has a vertical tangent line at a.

   (b) IF f has a vertical tangent line at a, THEN f is funky at a.

   Hint: Do Question 2 first.

4. Let f be a function defined, at least, on an interval of the form (c, ∞) for some $c \in \mathbb{R}$. Assume f is differentiable.

   Below are two claims. Are they true or false? If true, prove it. If false, provide a counterexample (and, as usual, show that your counterexample works).

   (a) IF f has a horizontal asymptote as $x \to \infty$, THEN $\lim_{x \to \infty} f'(x) = 0$.

   (b) IF $\lim_{x \to \infty} f'(x) = 0$, THEN f has a horizontal asymptote as $x \to \infty$.