

MAT 137Y: Calculus!

Problem Set 4 – Sample Solutions

1. Let $a, b > 0$. We want to study the curve with equation

$$(x^2 + y^2)^2 = ax^2 + by^2.$$

Notice that for each value of a and each value of b we get a different curve. You can see the graph on <http://tinyurl.com/mat137ps4>. (Strictly speaking we should also add the single point $(0, 0)$ to the graph you see at that url. For the purpose of this question, ignore the point $(0, 0)$.) You will find two sliders that allow you to change the values of a and b and see what happens to the graph.

- (a) First, let's fix $a = 6$ and $b = 1$. Prove that the curve has exactly 6 points with a horizontal tangent line and find their coordinates.

Hint: Use implicit differentiation.

We are looking for points on the curve

$$(x^2 + y^2)^2 = 6x^2 + y^2 \tag{1}$$

such that $\frac{dy}{dx} = 0$ (we see y as a function of x locally around any point of the curve).

Hence we have to solve the following system (i.e. we are looking for points (x, y) satisfying simultaneously both following equations),

$$\begin{cases} (x^2 + y^2)^2 = 6x^2 + y^2 \\ \frac{dy}{dx} = 0 \end{cases}$$

The first equation ensures that we are looking at points on the curve, and the second that the tangent is horizontal at this point.

By differentiating (1) with respect to x , we obtain

$$2 \left(2x + 2y \frac{dy}{dx} \right) (x^2 + y^2) = 12x + 2y \frac{dy}{dx}$$

Therefore, $\frac{dy}{dx} = 0$ is equivalent to

$$4x(x^2 + y^2) = 12x$$

or, equivalently,

$$x(x^2 + y^2) = 3x \tag{2}$$

You may be tempted to “cancel out the x ” on both sides of (2), but that is only possible when $x \neq 0$ (otherwise we are dividing by 0). Thus, we conclude that $\frac{dy}{dx} = 0$ if and only if $x = 0$ or $x^2 + y^2 = 3$.

- i. First case: Assume that $x = 0$.
Hence, by substituting $x = 0$ in (1), we obtain

$$\begin{aligned} y^4 &= y^2 \\ \Leftrightarrow y^2(y^2 - 1) &= 0 \\ \Leftrightarrow y^2 &= 1 \quad (\text{Indeed, we assumed that } (x, y) \neq (0, 0) \text{ in the question}) \\ \Leftrightarrow y &= -1 \text{ or } y = 1 \end{aligned}$$

Hence, we obtained the points $(0, -1)$ and $(0, 1)$.

- ii. Second case: Assume that $x^2 + y^2 = 3$.
Then, by substituting $x^2 + y^2 = 3$ in (1), we obtain:

$$6x^2 + y^2 = (x^2 + y^2)^2 = 3^2 = 9$$

Therefore, we are left with the system of equations

$$\begin{cases} x^2 + y^2 = 3 \\ 6x^2 + y^2 = 9 \end{cases}$$

This system has solution $x^2 = \frac{6}{5}$ and $y^2 = \frac{9}{5}$.

Hence we obtained the four points $(\pm\sqrt{\frac{6}{5}}, \pm\sqrt{\frac{9}{5}})$

Conclusion: for $a = 6$ and $b = 1$, the curve admits exactly 6 points with a horizontal tangent line, which are

$$(0, -1), (0, 1), \left(-\sqrt{\frac{6}{5}}, -\sqrt{\frac{9}{5}}\right), \left(-\sqrt{\frac{6}{5}}, \sqrt{\frac{9}{5}}\right), \left(\sqrt{\frac{6}{5}}, -\sqrt{\frac{9}{5}}\right), \left(\sqrt{\frac{6}{5}}, \sqrt{\frac{9}{5}}\right)$$

- (b) Keep $a = 6$ and $b = 1$. Prove that the curve has exactly 2 points with a vertical tangent line and find their coordinates. Notice that it is not enough to find these points: we also want you to prove algebraically that it has no others.

Hint: Use implicit differentiation, thinking of x as a function of y .

Similarly to the above question, we are looking for points (x, y) such that

$$\begin{cases} (x^2 + y^2)^2 = 6x^2 + y^2 \\ \frac{dx}{dy} = 0 \end{cases}$$

By differentiating (1) with respect to y , we obtain

$$2 \left(2x \frac{dx}{dy} + 2y \right) (x^2 + y^2) = 12x \frac{dx}{dy} + 2y.$$

Therefore, $\frac{dx}{dy} = 0$ is equivalent to

$$2y(x^2 + y^2) = y.$$

This is true if and only if $y = 0$ or $x^2 + y^2 = \frac{1}{2}$.

i. First case: $y = 0$.

By substituting $y = 0$ in (1), we obtain

$$\begin{aligned} x^4 &= 6x^2 \\ \Leftrightarrow x^2(x^2 - 6) &= 0 \\ \Leftrightarrow x^2 &= 6 \quad (\text{Indeed, we assumed that } (x, y) \neq (0, 0) \text{ in the question}) \end{aligned}$$

So we obtained two points $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

ii. Second case: $x^2 + y^2 = \frac{1}{2}$.

We substitute $x^2 + y^2 = \frac{1}{2}$ in (1) and we obtain $\frac{1}{4} = 6x^2 + y^2$. Thus, we are left with the system of equations

$$\begin{cases} x^2 + y^2 = \frac{1}{2} \\ 6x^2 + y^2 = \frac{1}{4} \end{cases}$$

This system has solution $x^2 = -\frac{1}{20}$ and $y^2 = \frac{11}{20}$.

This is a contradiction, since every real number x satisfies $x^2 \geq 0$.

Therefore, we conclude there is no point in the curve satisfying $x^2 + y^2 = \frac{1}{2}$.

Conclusion: for $a = 6$ and $b = 1$, the curve admits exactly 2 points with a vertical tangent line, which are $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

(c) Now play with the sliders and try different values of a and b . You will notice that sometimes the curve has exactly 2 points with a horizontal tangent line, and sometimes it has exactly 6 points with a horizontal tangent line. For which values of a and b does it have 6 and for which values does it have 2? Prove it.

We are going to prove that there are exactly 6 points with a horizontal tangent line when $a > 2b$ and exactly 2 points otherwise.

We want to find points (x, y) such that

$$\begin{cases} (x^2 + y^2)^2 = ax^2 + by^2 \\ \frac{dy}{dx} = 0 \end{cases}$$

for arbitrary $a, b > 0$.

By differentiating

$$(x^2 + y^2)^2 = ax^2 + by^2 \quad (3)$$

with respect to x , we obtain:

$$2 \left(2x + 2y \frac{dy}{dx} \right) (x^2 + y^2) = 2ax + 2by \frac{dy}{dx}$$

Thus, $\frac{dy}{dx} = 0$ is equivalent to

$$2x(x^2 + y^2) = ax,$$

which is equivalent to $x = 0$ or $x^2 + y^2 = \frac{a}{2}$.

i. First case: $x = 0$.

By substituting $x = 0$ in (3), we obtain

$$\begin{aligned} y^4 &= by^2 \\ \Leftrightarrow y^2(y^2 - b) &= 0 \\ \Leftrightarrow y^2 &= b \quad (\text{Indeed, we assumed that } (x, y) \neq (0, 0) \text{ in the question}) \\ \Leftrightarrow y &= -\sqrt{b} \text{ or } y = \sqrt{b} \end{aligned}$$

Hence the points $(0, -\sqrt{b})$ and $(0, \sqrt{b})$ admit a horizontal tangent line.

ii. Second case: $x^2 + y^2 = \frac{a}{2}$.

By substituting $x^2 + y^2 = \frac{a}{2}$ in (3), we obtain $ax^2 + by^2 = \frac{a^2}{4}$.

Thus, we get the system of equations

$$\begin{cases} x^2 + y^2 = \frac{a}{2} \\ ax^2 + by^2 = \frac{a^2}{4} \end{cases}$$

The second equation minus b times the first equation produces

$$(a - b)x^2 = \frac{a(a - 2b)}{4} \quad (4)$$

while the second equation minus a times the first equation produces

$$(a - b)y^2 = \frac{a^2}{4}. \quad (5)$$

If $a \leq b$, Equation (5) gives us a contradiction (remember that $a > 0$), so we do not get any new points with horizontal tangent line in that case. So let's assume $a > b$. Now let's look at equation (4).

- If $a = 2b$, then $x = 0$ and this case has already been treated.
- If $a < 2b$, then $x^2 < 0$ which is not possible, and, under this assumption this case doesn't produce new points with a horizontal tangent line either.
- If $a > 2b$, then we have 4 additional distinct points admitting a horizontal tangent line, whose coordinates are $\left(\pm\sqrt{\frac{a(a-2b)}{4(a-b)}}, \pm\frac{a}{2\sqrt{a-b}}\right)$.

Conclusion: the curve admits exactly 6 points with a horizontal line if $a > 2b$, and exactly 2 points with a horizontal line if $a \leq 2b$.

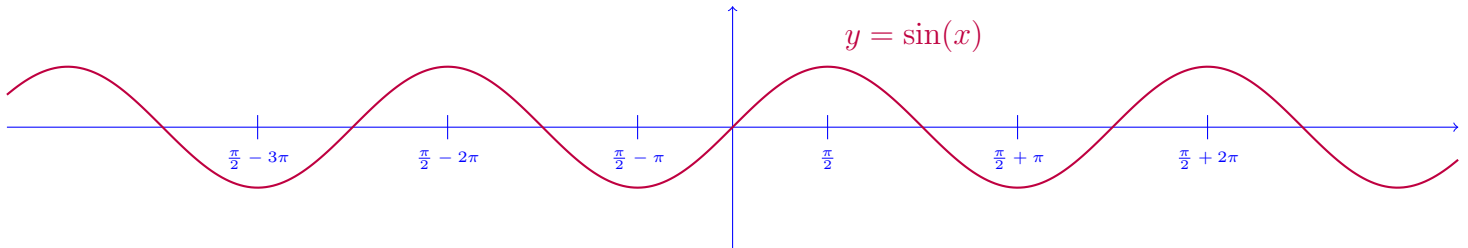
- (d) [**Do not submit.**] Repeat Question 1c for points with a vertical tangent line, instead of a horizontal tangent line.

Hint: You can use a symmetry argument and your solution to 1c to answer this question without having to do any calculations.

Points with vertical tangent line correspond to points with horizontal tangent line if we “swap x and y ”. This swap will reverse the roles of a and b .

The curve admits exactly 6 points with a vertical line if $b > 2a$, and exactly 2 points with a horizontal line if $b \leq 2a$.

2. Let us consider the function f defined by $f(x) = \sin x$. It is not one-to-one. For each $a \in \mathbb{R}$ we defined I_a to be the largest interval containing a such that the restriction of f to I_a is one-to-one.



- (a) There are some values of $a \in \mathbb{R}$ for which the above definition does not make sense: the interval I_a is not well-defined. What are these values?

Short explanation

The function \sin is one-to-one when restricted to any interval of the form

$$\left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi \right] \quad (6)$$

but it stops being one-to-one if we pick any interval bigger than those.

- When a is one of the endpoints of one of these intervals, then it belongs to two of them: one to the left, and one to the right. Both are equally “large”, so none of them qualifies as *the* largest. Therefore I_a is not well-defined in that case.
- Otherwise, a is an interior point to exactly one interval of the form (6), and that interval is I_a .

For those who prefer a very formal proof

- During this question, I will use several times the following fact:

Let I be an interval. If there exists $n \in \mathbb{Z}$ such that $\frac{\pi}{2} + n\pi$ is an interior point of I , then \sin isn't one-to-one on I .

(*)

Proof: If there exists $n \in \mathbb{Z}$ such that $\frac{\pi}{2} + n\pi$ is an interior point of I , then there exists $\varepsilon > 0$ such that $\frac{\pi}{2} + n\pi - \varepsilon$ and $\frac{\pi}{2} + n\pi + \varepsilon$ are both in I . But

$$\sin\left(\frac{\pi}{2} + n\pi + \varepsilon\right) = (-1)^n \cos(\varepsilon) = (-1)^n \cos(-\varepsilon) = \sin\left(\frac{\pi}{2} + n\pi - \varepsilon\right)$$

So \sin isn't one-to-one on I . ■

- Let $a \in \mathbb{R}$.
 Assume that $a = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$.
 According to (*), a has to be an endpoint of I_a .
 Notice that \sin is one-to-one on $[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi]$ and on $[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi]$ by (strict) monotonicity of \sin on these intervals.
 Both of these intervals are of length π and can't be extended without losing the injectiveness because of (*).
 Hence I_a is not uniquely defined in this case.
- Assume that $a \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$.
 Let $n = \lfloor \frac{a}{\pi} - \frac{1}{2} \rfloor$ and let $I_a = [\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi]$.
 Then a is an interior point of I_a , \sin is one-to-one on I_a by strict monotonicity and I_a can't be extended without losing the injectiveness because of (*).
 Hence I_a is the largest interval containing a such that \sin is one-to-one on I_a .

Conclusion: the interval I_a is well defined if and only if $a \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$.
 And the I_a are exactly the intervals $[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi]$ for $n \in \mathbb{Z}$.

- (b) There may be different values of $a \in \mathbb{R}$ that produce the same interval I_a . What is the largest number of integers $a \in \mathbb{Z}$ such that they all produce the same interval I_a ? (If you think the answer is n , you need to find n such integers as an example, and justify why it is impossible to find more than n .)

The largest number of integers in an interval I_a is **4**. Let's prove it!

- $\frac{\pi}{2} + 3\pi \approx 10.995\dots$ and $\frac{\pi}{2} + 4\pi \approx 14.9137\dots$

Therefore, $\{11, 12, 13, 14\} \subseteq \left(\frac{\pi}{2} + 3\pi, \frac{\pi}{2} + 4\pi\right)$.

Therefore $I_{11} = I_{12} = I_{13} = I_{14}$.

I have proven that we can find an interval produced by at least 4 different integers.

- Assume by contradiction there exists an interval I_a produced by at least five distinct integers.

This implies the length of I_a must be at least 4.

This is a contradiction, since we know from the previous question that the length of such an interval is exactly π .

Thus, I have proven we cannot find an interval produced by 5 different integers.

This ends the proof.

- (c) Construct a set $A \subseteq \mathbb{R}$ such that any two different elements of A produce different intervals, and all possible interval I_a are produced by some element in A . We call this set a “*complete list of representatives*”.¹

Let $A = \{k\pi : k \in \mathbb{Z}\}$.

Now we can make three claims:

- i. Every element in A produces a well-defined interval. Indeed

$$I_{k\pi} = \left[k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2} \right]$$

using our analysis from Question 2a.

- ii. If $k \neq k'$ then $I_{k\pi} \neq I_{k'\pi}$.

We see this is also true because we have an explicit form for these intervals.

- iii. All intervals are produced by $k\pi$ for some $k \in \mathbb{Z}$.

Again, we know this is true because we have a complete list of all these intervals from Question 2a.

¹A more formal way to say this is that $\forall b \in \mathbb{R}$ for which the interval I_b is well defined, there exists exactly one $a \in A$ such that $I_a = I_b$.

- (d) For each $a \in A$, let us call \heartsuit_a the inverse function of the restriction of f to I_a . What are the domain and the range of \heartsuit_a ? Sketch its graph (labelling the axes properly).

Let $a \in A$. Then there exists $n \in \mathbb{Z}$ such that $a = n\pi$.

According to the previous questions, $I_a = [\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi]$.

- We know that the restriction of \sin to I_a is one-to-one.
- Moreover the range of this restriction is $[-1, 1]$.

Indeed, let $c \in [-1, 1]$.

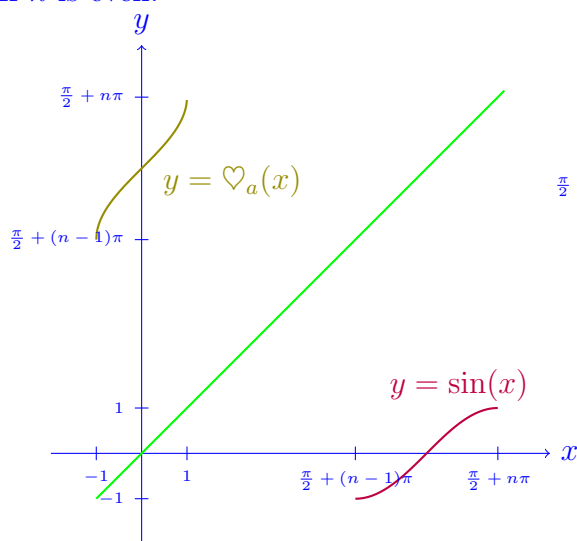
We know that

- \sin is continuous on $I_a = [\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi]$,
- if n is even then $\sin(\frac{\pi}{2} + (n-1)\pi) = -1$ and $\sin(\frac{\pi}{2} + n\pi) = 1$,
- if n is odd then $\sin(\frac{\pi}{2} + (n-1)\pi) = 1$ and $\sin(\frac{\pi}{2} + n\pi) = -1$.

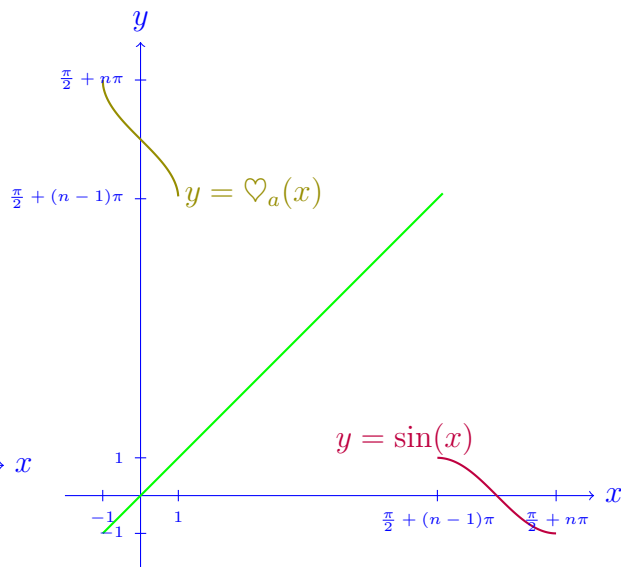
Hence, according to the IVT, there exists $x \in I_a$ such that $\sin(x) = c$.

Hence the restriction of \sin to I_a admits an inverse \heartsuit_a whose domain is $[-1, 1]$ and whose range is I_a .

If n is even:



If n is odd:



Bonus result

The following observation, if proven, can simplify the answers of the following questions!

Assume that $a = n\pi$ for $n \in \mathbb{Z}$, then

$$\heartsuit_a(x) = (-1)^n \arcsin(x) + n\pi \quad (7)$$

Indeed, define $g : [-1, 1] \rightarrow I_a$ by $g(x) = (-1)^n \arcsin(x) + n\pi$.

To prove that $g = \heartsuit_a$, I need to show that g is the inverse of the restriction of \sin to the interval I_a . This is equivalent to proving that

- $\forall x \in [-1, 1], \sin(g(x)) = x$
- $\forall x \in I_a, g(\sin x) = x$.

Let's prove the first identity. Let $x \in [-1, 1]$ Then

$$\begin{aligned} \sin(g(x)) &= \sin((-1)^n \arcsin(x) + n\pi) \\ &= (-1)^n \sin((-1)^n \arcsin(x)) \\ &= (-1)^{2n} \sin(\arcsin(x)) \\ &= \sin(\arcsin(x)) \\ &= x \end{aligned}$$

And let's prove the second identity. Let $x \in I_a = \left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$. Then

$$\begin{aligned} g(\sin(x)) &= (-1)^n \arcsin(\sin(x)) + n\pi \\ &= \arcsin((-1)^n \sin(x)) + n\pi \\ &= \arcsin(\sin(x - n\pi)) + n\pi \\ &= x - n\pi + n\pi \quad \text{since } x - n\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \\ &= x \end{aligned}$$

■

(e) For each $a \in A$, compute $\heartsuit_a(f(2018))$ and $f(\heartsuit_a(2018))$. Or, if they are not defined, explain why.

- Notice that $2018 \notin [-1, 1]$ so $\forall a \in A$, $f(\heartsuit_a(2018))$ isn't defined.
- Let $a \in A$. There exists $n \in \mathbb{Z}$ such that $a = n\pi$.
We are going to prove that

$$\heartsuit_a(\sin(2018)) = \begin{cases} 2018 - 642\pi + n\pi & \text{if } n \text{ is even} \\ -2018 + 642\pi + n\pi & \text{if } n \text{ is odd} \end{cases}$$

This can be summarized as $\heartsuit_a(\sin(2018)) = (-1)^n(2018 - 642\pi) + n\pi$.

– **Method 1:**

Notice that

$$\heartsuit_a(\sin(2018)) = x \iff \begin{cases} x \in I_a \\ \sin(2018) = \sin x \end{cases}$$

Let us call $x = (-1)^n(2018 - 642\pi) + n\pi$. I will show that $x \in I_a$ and $\sin x = \sin(2018)$ and that will complete the proof.

* For the first condition, notice that $2018 - 642\pi \approx 1.097$.

$$\text{Therefore, } \pm(2018 - 642\pi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

$$\text{Therefore, } x = (-1)^n(2018 - 642\pi) + n\pi \in I_a.$$

* For the second condition

$$\begin{aligned} \sin x &= \sin((-1)^n(2018 - 642\pi) + n\pi) \\ &= (-1)^n \sin((-1)^n(2018 - 642\pi)) \\ &= (-1)^n (-1)^n \sin(2018 - 642\pi) \\ &= \sin(2018 - 642\pi) = \sin(2018) \end{aligned}$$

– **Method 2,** using (7)

$$\begin{aligned} \heartsuit_a(\sin(2018)) &= (-1)^n \arcsin(\sin(2018)) + n\pi \\ &= (-1)^n(2018 - 642\pi) + n\pi \end{aligned}$$

(f) For each $a \in A$, derive an explicit formula for \heartsuit'_a .

Hint: Imitate the derivation in Video 4.7. Notice that it is not exactly the same.

Let $a \in A$. There exists $n \in \mathbb{Z}$ such that $a = n\pi$.

- **Method 1:**

For all $x \in (-1, 1)$, we have $\sin(\heartsuit_a(x)) = x$.

By differentiating with respect to x , we obtain:

$$\begin{aligned}\heartsuit'_a(x) \sin'(\heartsuit_a(x)) &= 1 \\ \Leftrightarrow \heartsuit'_a(x) \cos(\heartsuit_a(x)) &= 1 \\ \Leftrightarrow \heartsuit'_a(x) &= \frac{1}{\cos(\heartsuit_a(x))}\end{aligned}$$

Notice that we need to work with the open interval $(-1, 1)$ since $\cos(\heartsuit_a(x)) = 0$ when $x = -1$ or 1 .

We derive from $\cos^2(\heartsuit_a(x)) + \sin^2(\heartsuit_a(x)) = 1$ that

$$\cos(\heartsuit_a(x)) = \pm\sqrt{1 - \sin^2(\heartsuit_a(x))} = \pm\sqrt{1 - x^2}$$

If n is even, then \cos is positive on $(\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi)$ so that $\cos(\heartsuit_a(x)) = \sqrt{1 - x^2}$.

Otherwise, if n is odd, then \cos is negative on $(\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi)$ so that $\cos(\heartsuit_a(x)) = -\sqrt{1 - x^2}$.

We can summarize this by writing $\cos(\heartsuit_a(x)) = (-1)^n \sqrt{1 - x^2}$, and

$$\heartsuit'_a(x) = \frac{(-1)^n}{\sqrt{1 - x^2}}$$

- **Method 2,** using (7)

$$\heartsuit'_a(x) = \frac{d}{dx} [(-1)^n \arcsin(x) + n\pi] = (-1)^n \arcsin'(x) = \frac{(-1)^n}{\sqrt{1 - x^2}}.$$