MAT 137Y: Calculus!
Problem Set 4
Due on Thursday, November 22 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about implicit differentiation, inverse functions, and inverse trigonometric functions (Playlists 3 and 4).

1. Let $a, b > 0$. We want to study the curve with equation

$$(x^2 + y^2)^2 = ax^2 + by^2.$$

Notice that for each value of a and each value of b we get a different curve. You can see the graph on `http://tinyurl.com/mat137ps4` . (Strictly speaking we should also add the single point $(0, 0)$ to the graph you see at that url. For the purpose of this question, ignore the point $(0, 0)$.) You will find two sliders that allow you to change the values of a and b and see what happens to the graph.

(a) First, let's fix $a = 6$ and $b = 1$. Prove that the curve has exactly 6 points with a horizontal tangent line and find their coordinates.

Hint: Use implicit differentiation.

(b) Keep $a = 6$ and $b = 1$. Prove that the curve has exactly 2 points with a vertical tangent line and find their coordinates. Notice that it is not enough to find these points: we also want you to prove algebraically that it has no others.

Hint: Use implicit differentiation, thinking of x as a function of y.

(c) Now play with the sliders and try different values of a and b. You will notice that sometimes the curve has exactly 2 points with a horizontal tangent line, and sometimes it has exactly 6 points with a horizontal tangent line. For which values of a and b does it have 6 and for which values does it have 2? Prove it.

(d) ***[Do not submit.]*** Repeat Question 1c for points with a vertical tangent line, instead of a horizontal tangent line.

Hint: You can use a symmetry argument and your solution to 1c to answer this question without having to do any calculations.

2. Let us consider the function f defined by $f(x) = \sin x$. It is not one-to-one. For each $a \in \mathbb{R}$ we defined I_a to be the largest interval containing a such that the restriction of f to I_a is one-to-one.

   (a) There are some values of $a \in \mathbb{R}$ for which the above definition does not make sense: the interval I_a is not well-defined. What are these values?

   (b) There may be different values of $a \in \mathbb{R}$ that produce the same interval I_a. What is the largest number of integers $a \in \mathbb{Z}$ such that they all produce the same interval I_a? (If you think the answer is n, you need to find n such integers as an example, and justify why it is impossible to find more than n.)

   (c) Construct a set $A \subseteq \mathbb{R}$ such that any two different elements of A produce different intervals, and all possible interval I_a are produced by some element in A. We call this set a *"complete list of representatives"*.[1]

   (d) For each $a \in A$, let us call $\heartsuit_a$ the inverse function of the restriction of f to I_a. What are the domain and the range of $\heartsuit_a$? Sketch its graph (labelling the axes properly).

   (e) For each $a \in A$, compute $\heartsuit_a(f(2018))$ and $f(\heartsuit_a(2018))$. Or, if they are not defined, explain why.

   (f) For each $a \in A$, derive an explicit formula for $\heartsuit'_a$.
   Hint: Imitate the derivation in Video 4.7. Notice that it is not exactly the same.

[1] A more formal way to say this is that $\forall b \in \mathbb{R}$ for which the interval I_b is well defined, there exists exactly one $a \in A$ such that $I_a = I_b$.