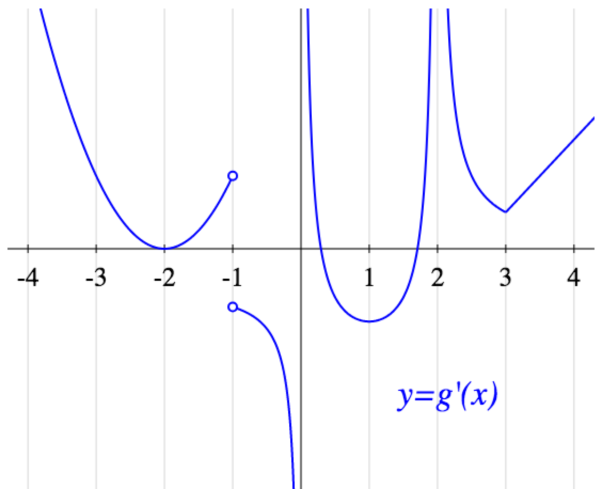
# MAT 137Y: Calculus!
## Problem Set 3
## Sample solutions

1. We know the function $g$ has domain $\mathbb{R}$ and is continuous everywhere. We also know that $g(0) = 0$. Here is the graph of its derivative:

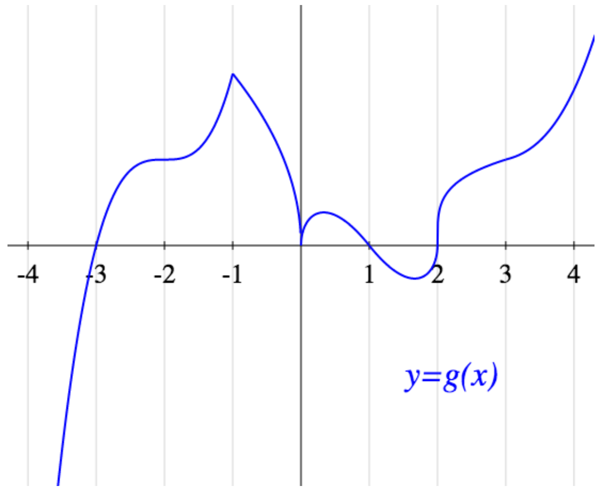



Sketch the graph of $g$.

The sketch will be approximate (we do not even have a scale on the $y$-axis) but there are a few important things your graph should show:

- $g(0) = 0$ (other $y$-values may differ significantly between your answer and this answer)
- horizontal tangent line at $x = -2$, but positive slope at both sides of it
- corner at $x = -1$
- cusp at $x = 0$
- local maximum at about $x = 0.3$
- local minimum at about $x = 1.7$
- vertical tangent line at $x = 2$
- $g$ should be defined and continuous everywhere
- the graph should be smooth (differentiable) everywhere else, including at $x = 3$.

Below is one possible sketch for the graph of $g$.



2. Consider the function $h$ given by the equation

$$h(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x+1}}}}}}}$$

where there are 2018 square roots in total. Find the equation of the line tangent to the graph of $h$ at the point with $x$–coordinate 0.

For every positive integer $n$, we define

$$h_n(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x+1}}}}}}},$$

where there are $n$ square roots. Notice that the original function is $h(x) = h_{2018}(x)$.

We are going to prove by induction that for every $n$, $h_n'(0) = 1 - \dfrac{1}{2^n}$.

- Base case: for $n = 1$, we need to show that $h_1'(0) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$.

  Since $h_1(x) = \sqrt{x+1}$, we obtain that $h_1'(x) = \dfrac{1}{2\sqrt{1+x}}$ so that $h_1'(0) = \dfrac{1}{2}$.

- Induction step: Let $n$ be a positive integer.

  Assume that $h_n'(0) = 1 - \dfrac{1}{2^n}$. We want to show that $h_{n+1}'(0) = 1 - \dfrac{1}{2^{n+1}}$.

  Since $h_{n+1}(x) = \sqrt{x + h_n(x)}$, we derive from the chain rule that

$$h_{n+1}'(x) = \frac{1 + h_n'(x)}{2\sqrt{x + h_n(x)}}.$$

  Hence $h_{n+1}'(0) = \dfrac{1 + h_n'(0)}{2\sqrt{0 + h_n(0)}} = \dfrac{1 + h_n'(0)}{2} = \dfrac{2 - \frac{1}{2^n}}{2} = \dfrac{2^{n+1} - 1}{2^{n+1}} = 1 - \dfrac{1}{2^{n+1}}$.

  This ends the induction proof.

According to the above formula, $h'(0) = h_{2018}'(0) = 1 - \dfrac{1}{2^{2018}}$. In addition, $h(0) = 1$.

Hence the tangent to the graph of $h$ at $(0,1)$ has the following equation:

$$y = \left(1 - \frac{1}{2^{2018}}\right)x + 1$$

3. The most common way to derive formulas for the derivatives of the six trig functions is the one you learned in the videos/class: we obtain the derivative of sin and cos from the definition ("the long way") and then we use the quotient rule to derive the rest. But we could have done it in other ways.

   For the purpose of this problem, assume you know the basic differentiation rules (linearity, power, product, quotient, and chain) but that you do not know yet any of the formulas for derivatives of trig functions.

   (a) Obtain a formula for the derivative of tan directly from the definition of derivative as a limit.

   *Hint:* Write $\tan x = \dfrac{\sin x}{\cos x}$ and use the formulas for the sine of the sum and the cosine of the sum. This is similar to the derivation in Video 3.11.

   Solution 1

   For any values of $x$ and $h$ for which $\tan(x)$ and $\tan(x+h)$ are defined:

$$\begin{aligned}
\frac{\tan(x+h)-\tan x}{h} &= \frac{\dfrac{\sin(x+h)}{\cos(x+h)} - \dfrac{\sin x}{\cos x}}{h} = \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x\cos(x+h)} \\
&= \frac{[\sin x\cos h + \cos x \sin h]\cos x - [\cos x\cos h - \sin x \sin h]\sin x}{h\cos x\cos(x+h)} \\
&= \frac{[\sin x\cos x - \cos x\sin x]\cos h + \left[\cos^2 x + \sin^2 x\right]\sin h}{h\cos x\cos(x+h)} \\
&= \frac{\sin h}{h\cos x\cos(x+h)}
\end{aligned}$$

   Notice that the above is well defined as long as $x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi \ : \ n \in \mathbb{Z}\right\}$ (which is the domain of tan) and $h$ is close to 0 but not 0.
   Therefore

$$\begin{aligned}
\tan'(x) &= \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} = \left[\lim_{h\to 0}\frac{\sin h}{h}\right]\frac{1}{\cos x}\left[\lim_{h\to 0}\frac{1}{\cos(x+h)}\right] \\
&= 1\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos x} = \frac{1}{\cos^2 x}
\end{aligned}$$

   In the last calculation, I have used that $\lim_{h\to 0}\dfrac{\sin h}{h} = 1$. In addition, I have used that, **since the function** cos **is continuous**

$$\lim_{h\to 0}\cos(x+h) = \cos x.$$

   Hence tan is differentiable on its domain and $\tan'(x) = \dfrac{1}{\cos^2 x} = \sec^2(x) = 1 + \tan^2 x.$

Solution 2

We can begin the same calculation but avoid some of the algebra if we apply the identity for the sum of the difference

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

to the numerator. Specifically:

$$\begin{aligned}
\frac{\tan(x+h)-\tan x}{h} &= \frac{\dfrac{\sin(x+h)}{\cos(x+h)} - \dfrac{\sin x}{\cos x}}{h} = \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x \cos(x+h)} \\
&= \frac{\sin[(x+h)-x]}{h\cos x\cos(x+h)} \\
&= \frac{\sin h}{h\cos x\cos(x+h)}
\end{aligned}$$

The continue from here as in Solution #1.

Solution 3

First notice that for $a, b \in \mathbb{R}$ such that $\tan(a+b)$, $\tan(a)$ and $\tan(b)$ are well defined, we have

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\cos(a)\cos(b)-\sin(a)\sin(b)} \\
&= \frac{\dfrac{\sin(a)}{\cos(a)} + \dfrac{\sin(b)}{\cos(b)}}{1 - \dfrac{\sin(a)}{\cos(a)}\cdot\dfrac{\sin(b)}{\cos(b)}} = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\tan(x+h)-\tan(x)}{h} &= \frac{\dfrac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)} - \tan(x)}{h} \\
&= \frac{\tan(x)+\tan(h)-\tan(x)+\tan^2(x)\tan(h)}{h(1-\tan(x)\tan(h))} \\
&= \frac{\tan(h)+\tan^2(x)\tan(h)}{h(1-\tan(x)\tan(h))} \\
&= \frac{1}{1-\tan(x)\tan(h)}\left(\frac{\tan(h)}{h} + \tan^2(x)\cdot\frac{\tan(h)}{h}\right)
\end{aligned}$$

Notice that the above is well defined as long as $x$ is in the domain of tan and $h$ is close to 0 but not 0.

Notice also that

$$\lim_{h\to 0}\frac{\tan(h)}{h} = \lim_{h\to 0}\left(\frac{1}{\cos(h)}\cdot\frac{\sin(h)}{h}\right) = \lim_{h\to 0}\frac{1}{\cos(h)}\cdot\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

Finally, notice that **since the function** tan **is continuous**, we have

$$\lim_{h\to 0} \tan h = \tan 0 = 0$$

Putting it all together:

$$\begin{aligned}\lim_{h\to 0} \frac{\tan(x+h)-\tan(x)}{h} &= \lim_{h\to 0} \left(\frac{1}{1-\tan(x)\tan(h)}\left(\frac{\tan(h)}{h} + \tan^2(x)\cdot\frac{\tan(h)}{h}\right)\right) \\ &= 1+\tan^2(x)\end{aligned}$$

Hence tan is differentiable on its domainand $\tan'(x) = 1+\tan^2(x)$.

(b) Use your answer to Question 3a and implicit differentiation on

$$\sec^2 x = 1+\tan^2 x$$

to obtain a formula for the derivative of sec.

By taking the derivative with respect to $x$ in the above identity, we obtain:

$$\begin{aligned}\frac{d}{dx}\left[\sec^2 x\right] &= \frac{d}{dx}\left[1+\tan^2 x\right] \\ 2\sec x \sec' x &= 2\tan x \tan' x \\ 2\sec x \sec' x &= 2\tan x \sec^2 x\end{aligned}$$

Furthermore, since $\sec(x)\neq 0$, we get

$$\sec'(x) = \tan(x)\sec(x).$$

(c) Use your answer to Question 3b to obtain a formula for the derivative of cos.

Since $\cos(x) = \dfrac{1}{\sec(x)}$, we obtain

$$\begin{aligned}\cos'(x) &= -\frac{\sec'(x)}{\sec^2(x)} = -\frac{\tan(x)\sec(x)}{\sec^2(x)} \\ &= -\frac{\tan(x)}{\sec(x)} = -\tan(x)\cos(x) = -\sin(x)\end{aligned}$$

We got

$$\cos'(x) = -\sin(x)$$

*Note: The above proof only works for* $x\in\mathbb{R}\setminus\left\{\frac{\pi}{2}+n\pi \,:\, n\in\mathbb{Z}\right\}$*. We can prove that it works on all of* $\mathbb{R}$*, but we won't do it here.*

(d) Use your answer to Question 3c and the equations

$$\cos x = \sin\left(\frac{\pi}{2} - x\right), \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

to obtain a formula for the derivative of sin.

We derive from the identity $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$ and the chain rule that

$$\sin'(x) = -\cos'\left(\frac{\pi}{2} - x\right)$$

Hence, using Question 3c, we obtain

$$\sin'(x) = \sin\left(\frac{\pi}{2} - x\right)$$

Finally,

$$\sin'(x) = \cos(x)$$

4. Let $f$ be a continuous function with domain $\mathbb{R}$. Assume that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Prove that $f$ takes all possible real values. In other words, prove that for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

   *Hint:* As part of your proof, you will need to use the IVT, the definition of $\lim_{x\to\infty} f(x) = \infty$, and the definition of $\lim_{x\to-\infty} f(x) = -\infty$. If you do not use the three of them (or something related), your proof is probably wrong.

   WTS: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y$.

   - Let $y \in \mathbb{R}$.
   - Since $\lim_{x\to-\infty} f(x) = -\infty$, using $y$ as the bound in the definition of this limit, we know there exists $M \in \mathbb{R}$ such that
   $$\forall x \in \mathbb{R}, (x < M \implies f(x) < y) \tag{1}$$
   - Since $\lim_{x\to+\infty} f(x) = +\infty$, using $y$ as the bound in the definition of this limit, we know there exists $N \in \mathbb{R}$ such that
   $$\forall x \in \mathbb{R}, (x > N \implies f(x) > y) \tag{2}$$
   - We must have $M < N$ since otherwise we could get a contradiction between (1) and (2).
   - I want to use the IVT for the function $f$ on the interval $[M-1, N+1]$. First, we need to check the hypotheses. We know that
     (i) $f$ is continuous on $[M-1, N+1]$ (because $f$ is continuous on $\mathbb{R}$),
     (ii) $f(M-1) < y$ by (1), and
     (iii) $f(N+1) > y$ by (2).

     Hence, according to the IVT, there exists $x \in [M-1, N+1] \subset \mathbb{R}$ such that $f(x) = y$. That is what we wanted to show.