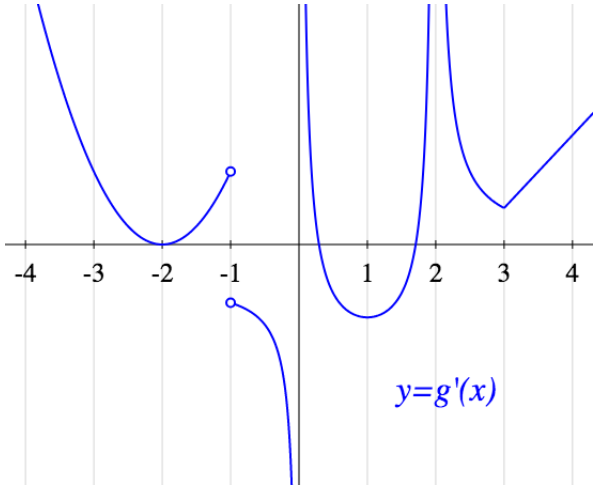
MAT 137Y: Calculus!
Problem Set 3
Due on Thursday, November 1 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about the Value Theorems and the definition and basic properties of derivatives (Videos 2.21 to 3.12)

1. We know the function g has domain $\mathbb{R}$ and is continuous everywhere. We also know that $g(0) = 0$. Here is the graph of its derivative:



Sketch the graph of g.

2. Consider the function h given by the equation

$$h(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x+1}}}}}}}$$

where there are 2018 square roots in total. Find the equation of the line tangent to the graph of h at the point with x–coordinate 0.

3. The most common way to derive formulas for the derivatives of the six trig functions is the one you learned in the videos/class: we obtain the derivative of sin and cos from the definition ("the long way") and then we use the quotient rule to derive the rest. But we could have done it in other ways.

   For the purpose of this problem, assume you know the basic differentiation rules (linearity, power, product, quotient, and chain) but that you do not know yet any of the formulas for derivatives of trig functions.

   (a) Obtain a formula for the derivative of tan directly from the definition of derivative as a limit.

   Hint: Write $\tan x = \dfrac{\sin x}{\cos x}$ and use the formulas for the sine of the sum and the cosine of the sum. This is similar to the derivation in Video 3.11.

   (b) Use your answer to Question 3a and implicit differentiation on

   $$\sec^2 x = 1 + \tan^2 x$$

   to obtain a formula for the derivative of sec.

   (c) Use your answer to Question 3b to obtain a formula for the derivative of cos.

   (d) Use your answer to Question 3c and the equations

   $$\cos x = \sin\left(\frac{\pi}{2} - x\right), \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

   to obtain a formula for the derivative of sin.

4. Let f be a continuous function with domain $\mathbb{R}$. Assume that $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$. Prove that f takes all possible real values. In other words, prove that for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

   Hint: As part of your proof, you will need to use the IVT, the definition of $\lim\limits_{x\to\infty} f(x) = \infty$, and the definition of $\lim\limits_{x\to-\infty} f(x) = -\infty$. If you do not use the three of them (or something related), your proof is probably wrong.