MAT 137Y: Calculus!
Problem Set 2 – Sample Solutions

1. Sketch the graph of a function f that satisfies all 12 conditions below simultaneously. For this question only, you do not need to prove or explain your answer, as long as the graph is correct and very clear. If you cannot satisfy all the properties at once, get as many as you can.

   (a) The domain of f is, at least, $(-5, 5)$

   (b) $\lim_{x \to a} f(x)$ exists for every a in the domain of f, except $a = 0$.

   (c) $\lim_{x \to 0} f(x)$ DNE

   (d) $\lim_{x \to 1} f(x) = 0$

   (e) $\lim_{x \to -1} f(x) = 0$

   (f) $\lim_{x \to 3} f(x) = 0$

   (g) $\lim_{x \to -3} f(x) = 0$

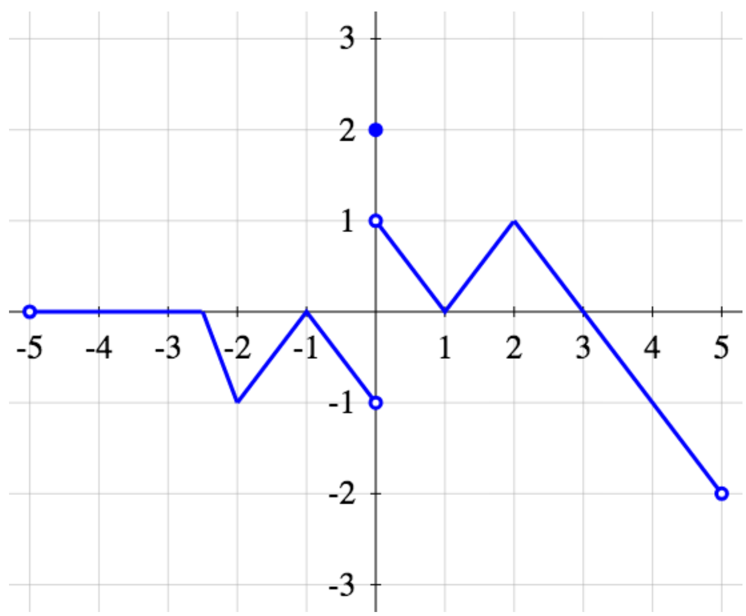
   (h) $\lim_{x \to 0} f(f(x)) = 0$

   (i) $\lim_{x \to 1} f(f(x)) = 1$

   (j) $\lim_{x \to -1} f(f(x)) = -1$

   (k) $\lim_{x \to 3} f(f(x))$ DNE

   (l) $\lim_{x \to -3} f(f(x)) = 2$



2. Let $a \in \mathbb{R}$. Let f and g be functions defined at least on an interval centered at a, except possibly at a. Is each of the following claims true or false? If it is false, show it with a counterexample. If it is true, prove it. (The proof should be a short, "one-line" proof using the properties of limits you already know. Do not use the formal definition of limit. No epsilons allowed in this question.)

   (a) IF $\lim_{x\to a}[f(x)+g(x)]$ exists and $\lim_{x\to a} f(x)$ exists, THEN $\lim_{x\to a} g(x)$ exists.
   This statement is TRUE. Let's prove it.
   Since $\lim_{x\to a}[f(x)+g(x)]$ and $\lim_{x\to a} f(x)$ both exist, the limit laws tells us that $g(x) = \big(f(x)+g(x)\big) - f(x)$ also has a limit as x approaches a. Specifically:

$$\lim_{x\to a} g(x) \;=\; \lim_{x\to a}[(f(x)+g(x)) - f(x)] \;=\; \left[\lim_{x\to a}(f(x)+g(x))\right] \;-\; \left[\lim_{x\to a} f(x)\right]$$

   (b) IF $\lim_{x\to a}[f(x)\cdot g(x)]$ exists and $\lim_{x\to a} f(x)$ exists, THEN $\lim_{x\to a} g(x)$ exists.
   This statement is FALSE. Below is a counter-example.
   - Take $f(x) = x^2$ and $g(x) = \frac{1}{x}$ defined on $(-\infty, 0)\cup(0,+\infty)$.
   - Then $\lim_{x\to 0} f(x)\cdot g(x) = \lim_{x\to 0} x = 0$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$,
   - But $\lim_{x\to 0} g(x) = \lim_{x\to 0}\dfrac{1}{x}$ doesn't exist.

3. Prove, directly from the formal definition of limit, that

$$\lim_{x \to 1} \left(x^3 + 2x\right) = 3.$$

Write a proof directly from the definition. Do not use any of the limit laws.

WTS: $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbb{R},\ \big(0 < |x - 1| < \delta \implies |x^3 + 2x - 3| < \varepsilon\big)$

We present two different proofs (among many) with different suitable δ.

Proof #1:

- Let $\varepsilon > 0$.
- Take $\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$.
- Let $x \in \mathbb{R}$.
- Assume $0 < |x - 1| < \delta$.
- Notice that

$$\begin{aligned} |x-1| < \delta \leq 1 &\implies -1 < x - 1 < 1 \\ &\implies 0 < x < 2 \\ &\implies \begin{cases} 0 < x^2 < 4 \\ 0 < x < 2 \end{cases} \\ &\implies 0 < x^2 + x + 3 < 9 \\ &\implies |x^2 + x + 3| < 9 \end{aligned}$$

and that $|x - 1| < \delta \leq \frac{\varepsilon}{9}$.
Thus

$$\begin{aligned} |x^3 + 2x - 3| &= |(x-1)(x^2 + x + 3)| \\ &= |x - 1| \cdot |x^2 + x + 3| \\ &< 9 \cdot \frac{\varepsilon}{9} \\ &= \varepsilon \end{aligned}$$

Hence $|x^3 + 2x - 3| < \varepsilon$.

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Proof #2:

- Let $\varepsilon > 0$.
- Take $\delta = \min\left\{ \sqrt[3]{\frac{\varepsilon}{3}}, \frac{\sqrt{\varepsilon}}{3}, \frac{\varepsilon}{15} \right\}$.
- Let $x \in \mathbb{R}$.
- Assume $0 < |x-1| < \delta$.
- Then,

$$\begin{aligned}
\left|x^3+2x-3\right| &= \left|(x-1)(x^2+x+3)\right| \\
&= \left|(x-1)((x-1)^2+3x+2)\right| \\
&= \left|(x-1)((x-1)^2+3(x-1)+5)\right| \\
&= \left|(x-1)^3+3(x-1)^2+5(x-1)\right| \\
&\le |x-1|^3+3|x-1|^2+5|x-1| \text{ by the Triangle Inequality} \\
&< \delta^3+3\delta^2+5\delta \\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Hence $|x^3+2x-3| < \varepsilon$.

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4. Let f be a function with domain $(-\infty, 0) \cup (0, \infty)$. Prove that

IF $\lim_{x \to 0} f(x) = \infty$
THEN $\lim_{x \to 0} f(x)$ does not exist

Notes: Before you write this proof, make sure you understand the precise definition of "the limit is ∞" and the definition of "the limit does not exist". Notice that the definition of "the limit does not exist" is *not* the negation of "the limit is L". If your definitions are not correct, then your proof cannot possibly be correct, and you won't get any credit. Make sure to write a formal proof directly from the formal definitions, without using any limit laws or similar properties.

Remember that:

(a) "$\lim_{x \to 0} f(x) = \infty$" means:

$$\forall M \in \mathbb{R},\ \exists \delta > 0,\ \forall x \in \mathbb{R},\ \big(0 < |x| < \delta \implies f(x) > M\big)$$

(b) "$\lim_{x \to 0} f(x)$ doesn't exist" means:

$$\forall L \in \mathbb{R},\ \exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x \in \mathbb{R},\ \big(0 < |x| < \delta \text{ and } |f(x) - L| \geq \varepsilon\big)$$

Proof of the statement:
We assume that $\lim_{x \to 0} f(x) = \infty$ and we want to show that $\lim_{x \to 0} f(x)$ doesn't exist.

- Let $L \in \mathbb{R}$.
- Take $\varepsilon = 1$. Notice that $\varepsilon > 0$.
  (Note: in this problem, any choice of ε would work, but we do need to choose one.)
- Let $\delta > 0$.
- Since $\lim_{x \to 0} f(x) = \infty$, (using $M = L + 1$ in the definition), there exists $\delta_1 > 0$ such that for every $x \in \mathbb{R}$,
  $$\text{if } 0 < |x| < \delta_1 \text{ then } f(x) > L + 1. \tag{1}$$
- Take $x = \min\left\{\frac{\delta}{2}, \frac{\delta_1}{2}\right\}$. Notice that $|x| = x$ and $x \neq 0$.
  - Then we have that $0 < |x| < \delta$.
  - In addition, since $0 < |x| = x < \delta_1$, from (1) we know that $f(x) > L + 1$. Therefore $f(x) - L > 1$ and thus $|f(x) - L| > 1 = \varepsilon$.
- We have verified that $0 < |x| < \delta$ and $|f(x) - L| \geq \varepsilon$, as we needed.

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