MAT 137Y: Calculus!
Problem Set 1
Sample Solutions

1. Negate the following statement without using any negative words ("no", "not", "none", "zero", etc.):

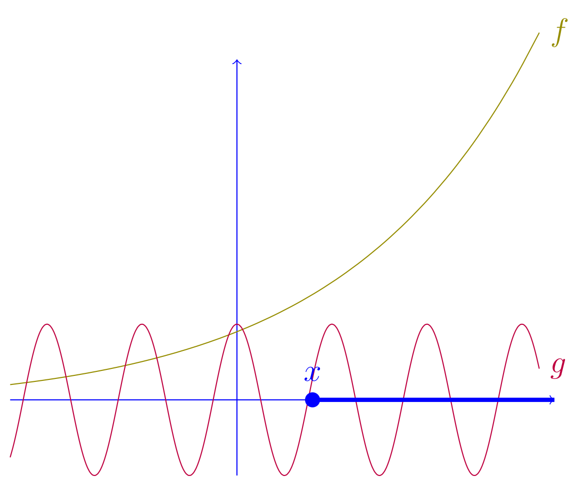
   > "A professor at a university in Ontario has written a book that has the property that every word in every even-numbered page begins with a letter that comes alphabetically before the letter it ends with."

   Every book written by any professor in any university in Ontario has a word in an even-numbered page whose first letter is its last letter or a letter that comes alphabetically after it.

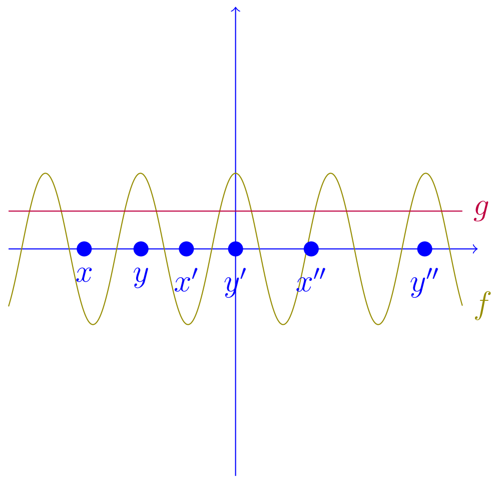
2. In this problem we will only consider (real-valued) functions with domain $\mathbb{R}$. We define two new concepts. Let f and g be two functions.

   - We say that f *is happier than* g when $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$,
   $$x < y \implies f(y) > g(y).$$
   - We say that f *is luckier than* g when $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t.
   $$x < y \textbf{ AND } f(y) > g(y).$$

   Geometrically, f is happier than g if the graph of f is above the graph of g for all points to the right of some fixed value x. For example:



And f is luckier than g, if for every x there exists at least one point y after x where the graph f is above the graph of g, as illustrated below.



Notice that the "value of y" depends on the "value of x". In other words, this means that the graph of f keeps crossing above the graph of g, but it does not necessarily stay permanently above it.

We also need a new piece of notation. Given any function f and any $t \in \mathbb{R}$, we define a new function, called f_t, via the equation

$$f_t(x) = f(x) + t$$

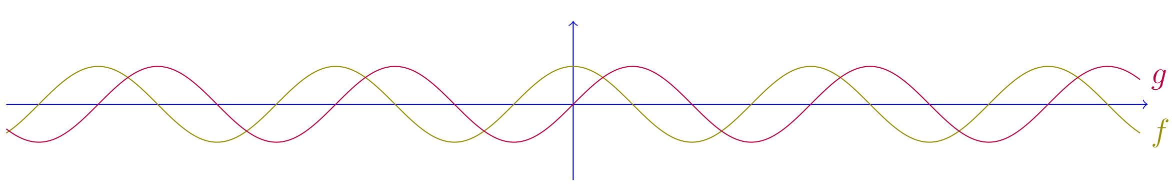
for all $x \in \mathbb{R}$.

Below are three claims. Which ones are true and which ones are false? If a claim is true, prove it. If a claim is false, show it with a counterexample.

(a) If f and g are two functions and f is luckier than g, then f is happier than g.

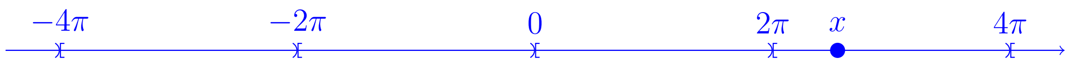
The statement is FALSE. Indeed, below is a counter-example.

Take $f(x) = \cos(x)$ and $g(x) = \sin(x)$. I claim that f is luckier than g but f is not happier than g. This is because the graph of f keeps crossing above the graph of g, but it never stays permanently above it.



We can also prove these two claims rigorously.

- Let's prove that f is luckier than g.
  - Fix an arbitrary $x \in \mathbb{R}$. I need to find one value of $y \in \mathbb{R}$ such that $x < y$ and $\cos y > \sin y$.
  - Notice there is a unique $k \in \mathbb{Z}$ such that $x \in [2k\pi, 2(k+1)\pi)$.



    I take $y = 2(k+1)\pi$.
  - Then $x < y$. In addition $\cos y = 1$ and $\sin y = 0$ so $f(y) = 1 > 0 = g(y)$.

- Next, let's prove that f is not happier than g. I will write the negation of "f is happier than g". I need to prove that

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t.} \quad [\, x < y \text{ and } f(y) \leq g(y)\,]$$

  Once I have it that way, notice that this proof is very similar to the previous one, swapping the roles of x and y (and replacing one '$<$' with '$\leq$'). I won't write the details.

(b) If f and g are two functions and f is happier than g, then f is luckier than g.

The statement is TRUE. Let's prove it.

- Let f and g be two functions. Assume that f is happier than g.
  We want to show that f is luckier than g.
- Let $x \in \mathbb{R}$. We need to find $y \in \mathbb{R}$ such that $y > x$ and $f(y) > g(y)$
- Since f is happier than g, we know there exists $x' \in \mathbb{R}$ such that for any $y \in \mathbb{R}$, if $y > x'$ then $f(y) > g(y)$.
- Now take $y = \max(x+1, x'+1)$.
  - By construction $y > x$.
  - In addition, since $y > x'$, we also know $f(y) > g(y)$.

Remark: Notice the structure of the proof. You want to show that f is luckier than g. First fix an arbitrary x, then construct y.

(c) For every function f there exists a function g such that for every $t \in \mathbb{R}$, g is happier than f_t.

The statement is TRUE. Let's prove it.

- Let f be a function with domain $\mathbb{R}$.
- I construct the function g by the equation $g(x) = f(x) + x$.
- Let $t \in \mathbb{R}$. We want to show that g is happier than f_t.
- Take $x = t$. We want to show that $\forall y \in \mathbb{R},\ y > x \implies g(y) > f_t(y)$.
- Indeed, let $y \in \mathbb{R}$ and assume that $y > x$. Then

$$g(y) = f(y) + y > f(y) + x = f(y) + t = f_t(y).$$

Remark: Notice the structure of the proof. First fix f, then construct g. You need to say what g is, which may depend on f, but not on t.

3. Prove by induction that for every positive integer n, the number $2^{3n} + 6$ is a multiple of 7.

- Base case: for $n = 1$.
  $2^{3\cdot 1} + 6 = 2^3 + 6 = 8 + 6 = 14 = 2 \cdot 7$.

- Inductive step: let n be a positive integer.
  Assume that $2^{3n} + 6$ is a multiple of 7, i.e. there exists $k \in \mathbb{N}$ such that $2^{3n} + 6 = 7k$.
  We want to show that $2^{3(n+1)} + 6$ is also a multiple of 7.

$$\begin{aligned}
2^{3(n+1)} + 6 &= 2^{3n+3} + 6 \\
&= 2^3 2^{3n} + 6 \\
&= 8 \cdot 2^{3n} + 6 \\
&= 8(2^{3n} + 6 - 6) + 6 \\
&= 8(2^{3n} + 6) - 8 \cdot 6 + 6 \\
&= 8(2^{3n} + 6) - 7 \cdot 6 \\
&= 8 \cdot 7 \cdot k - 7 \cdot 6 \qquad \textit{(Inductive hypothesis)} \\
&= 7(8k - 6)
\end{aligned}$$

which is, indeed, a multiple of 7.