Our textbook introduces exponential, logarithms, and their derivatives in a very different manner which will not make sense just yet. These notes are intended to replace sections 7.2–7.5 from the textbook. (You may read those sections from the book if you wish, although you may want to wait until we study integration and the Fundamental Theorem of Calculus.) The notes are a very short sketch and should be complemented with the lectures. The notes also contain some facts about exponentials and logarithms that you already learned in high school.

To complement the notes, do the practice problems from sections 7.2–7.5 that are posted on the course website.

1 Properties of exponentials

The expression $a^x$ makes sense whenever $a > 0$ and $x \in \mathbb{R}$. The following are common properties of exponentials that you already know. For every $a, b > 0$ and every $x, y \in \mathbb{R}$:

- $a^0 = 1$
- $a^1 = a$
- $a^{-x} = \frac{1}{a^x}$
- $a^x a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

When $a > 1$, the graph of $y = a^x$ looks like
When $0 < a < 1$, the graph of $y = a^x$ looks like

\begin{figure}
\centering
\includegraphics[width=\textwidth]{exp_graph.png}
\end{figure}

\section{Derivative of exponentials. Definition of $e$.}

Let us fix a constant $a > 0$. Consider the function $f(x) = a^x$. We want to compute the derivative of $f$ from the definition of derivative as a limit.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}
\]

Let us call $L_a = \lim_{h \to 0} \frac{a^h - 1}{h}$. If this limit exists, then $L_a$ simply represents a number. In that case, we have that

\[
\frac{d}{dx} a^x = L_a a^x
\]

In particular, $f'(0) = L_a$. We can try to compute this limit numerically. By plugging values of $h$ close to 0, it appears that

\[
L_2 = 0.6931\ldots, \quad L_3 = 1.0986\ldots
\]

We also see that $L_a$, as a function of $a$, is increasing.
**Definition.** We define $e$ as the only number $e > 0$ such that $L_e = 1$. In other words, $e$ is defined as the only number $e > 0$ such that $\frac{d}{dx} e^x = e^x$.

For any other base $a$, we will determine the value of $L_a$ in a little bit.

### 3 Definition of logarithm

Let $a > 0$, $a \neq 1$. Let $x > 0$. By definition, $\log_a x$ is the only number $y \in \mathbb{R}$ such that $a^y = x$. In other words:

$$\log_a x = y \iff a^y = x$$

For example, if we want to compute $\log_2 8$, we are looking for a number $y$ such that $2^y = 8$. The number $y = 3$ works, so $\log_2 8 = 3$.

For now, this is going to be our definition of logarithm. There is a better definition, but it requires integration (see section 7.2 of the textbook), so it will have to wait.

The definition means that logarithms are inverse functions of exponentials. This means that for every $a > 0$, $a \neq 1$, $x \in \mathbb{R}$ we have

$$\log_a a^x = x,$$

and for every $a > 0$, $a \neq 1$, $y > 0$, we have

$$a^{\log_a y} = y$$

The number $e$ is one special basis for logarithms. For every $x > 0$, we write $\ln x$ or $\log x$ for $\log_e x$.

**Note:** Historically, $\ln x$ meant $\log_e x$ and $\log x$ meant $\log_{10} x$. This comes from the day when people had to use logarithm tables to compute things by hand; both $\log_e x$ and $\log_{10} x$ were important and we used special names for both. Ever since the invention of the computer, there is no longer any need for ever using logarithms in base 10. Nowadays both $\ln x$ and $\log x$ mean $\log_e x$; mathematicians prefer to use $\log x$ whereas people in science prefer to use $\ln x$. As an exception, and mostly for historical reasons, chemists still use $\log x$ to mean $\log_{10} x$.

### 4 Properties of logarithms

When $a > 1$, the graph of $y = \log_a x$ looks like
Exponential and logarithms

When $0 < a < 1$, the graph of $y = \log_a x$ looks like

Here are some basic properties of logarithms. For every $a > 0$, $a \neq 1$, $x, y > 0$, $b > 0$, $b \neq 1$, $r \in \mathbb{R}$:

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a \frac{1}{x} = -\log_a x$
- $\log_a (xy) = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^r = r \log_a x$
- $\log_a x = \frac{\log_b x}{\log_b a}$

You should be able to prove every one of these properties from the definition of logarithm above, and from the properties of exponentials. We will do one as an example. We suggest you try the rest as an exercise.

**Theorem.** Let $a > 0$, $a \neq 1$. Let $x, y > 0$. Then $\log_a (xy) = \log_a x + \log_a y$. 
Proof. Let us call \( \log_a x = p \) and \( \log_a y = q \). By definition of logarithm this means \( a^p = x \) and \( a^q = y \). Now we can compute the following:

\[
a^{p+q} = a^p a^q = xy
\]

which, by definition of logarithm, means that \( \log_a(xy) = p + q = \log_a x + \log_a y \). \(\square\)

5 Derivative of exponentials. Part 2

We can use logarithms to rewrite an arbitrary exponential as an exponential with base \( e \). Specifically, for every \( a > 0, x \in \mathbb{R} \), we can write:

\[
a^x = e^{x \log a}
\]

(Why is this true?)

Let’s go back to finding the derivative of an arbitrary exponential function. Let us fix \( a > 0 \) and let \( f(x) = a^x \). Then

\[
f'(x) = \frac{d}{dx} a^x = \frac{d}{dx} e^{x \log a} = e^{x \log a} \frac{d}{dx} (x \log a) = e^{x \log a} \log a = a^x \log a
\]

where we have used the chain rule.

6 Derivatives of logarithms

Let us call \( g(x) = \ln x \). We want to find an expression for \( g'(x) \). We start by considering the identity:

\[
x = e^{\ln x}
\]

We can differentiate both sides with respect to \( x \):

\[
\frac{d}{dx} x = \frac{d}{dx} e^{\ln x}
\]

On the left-hand side, we get \( \frac{d}{dx} x = 1 \). On the right-hand side, using the chain rule, we get:

\[
\frac{d}{dx} e^{\ln x} = e^{\ln x} \frac{d}{dx} \ln x = x f'(x)
\]

so that we get \( 1 = xf'(x) \) or \( f'(x) = \frac{1}{x} \). In other words:

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

A similar calculation, left as an exercise to you, produces:

\[
\frac{d}{dx} \log_a x = \frac{1}{x \ln a}
\]
7 The power rule

You already know the power rule. If \( c \) is a constant:

\[
\frac{d}{dx} x^c = cx^{c-1}
\]

However, so far we were only able to prove it for certain values of \( c \). We can now prove it for arbitrary values of \( c \) (even for non-rational numbers!) We write \( x^c \) as an exponential with base \( e \), and then we use the chain rule:

\[
\frac{d}{dx} x^c = \frac{d}{dx} e^{c \ln x} = e^{c \ln x} \frac{d}{dx} (c \ln x) = e^{c \ln x} c \frac{1}{x} = x^c \frac{c}{x} = cx^{c-1}
\]

8 Logarithmic differentiation

Example 1. Consider the function \( f(x) = (x^2 + 1)^x \). We want to compute its derivative. There are at least two ways to do this.

- **Method 1**: Rewrite the function as an exponential with base \( e \).
  
  First, we rewrite \( f \) as
  
  \[
  f(x) = (x^2 + 1)^x = e^{x \ln(x^2 + 1)}
  \]
  
  Second, we use the chain rule:
  
  \[
  f'(x) = e^{x \ln(x^2 + 1)} \frac{d}{dx} [x \ln(x^2 + 1)]
  \]
  
  and the product rule:
  
  \[
  f'(x) = e^{x \ln(x^2 + 1)} \left[ \ln(x^2 + 1) + x \frac{2x}{x^2 + 1} \right] = (x^2 + 1)^x \left[ \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]
  \]

- **Method 2**: Logarithmic differentiation.
  
  Let us call \( y = (x^2 + 1)^x \). Taking logarithms on both sides, we get:
  
  \[
  \ln y = \ln(x^2 + 1)^x = x \ln(x^2 + 1)
  \]
  
  so that we can write:
  
  \[
  \frac{d}{dx} \ln y = \frac{d}{dx} [x \ln(x^2 + 1)]
  \]
  
  and we use implicit differentiation:
  
  \[
  \frac{1}{y} \frac{dy}{dx} = \ln(x^2 + 1) + x \frac{2x}{x^2 + 1} = \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}
  \]
  
  from where we get
  
  \[
  \frac{dy}{dx} = y \left[ \ln(x^2 + 1) + x \frac{2x}{x^2 + 1} \right]
  \]

Example 2 Consider the function \( g(x) = \frac{\sqrt{x^2 + 3x}}{(1 + \sin^2 x)(x^2 + 1)^5} \). We want to compute \( g'(x) \).
• **Method 1.** We do not need any new method for computing this derivative. Just use the quotient rule, the product rule, the power rule, and the chain rule repeatedly, and you will get the result. However, this is long, tedious, and very error-prone. The second method is much more efficient.

• **Method 2.** Call $y = g(x)$. We can write

$$\ln y = \frac{1}{2} \ln(x^2 + 3x) - \ln(1 + \sin^2 x) - 5\ln(x^2 + 1)$$

so that

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left[ \frac{1}{2} \ln(x^2 + 3x) - \ln(1 + \sin^2 x) - 5\ln(x^2 + 1) \right]$$

and a much simpler calculation produces

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \frac{2x + 3}{x^2 + 3x} - \frac{2\sin x \cos x}{1 + \sin^2 x} - 5 \frac{2x}{x^2 + 1}$$

from where we can get an expression for $\frac{dy}{dx}$. 
