MAT 137Y: Calculus!
Problem Set B.

This problem set is intended to help you prepare for Test #2. It is not comprehensive: it only contains problems from some sections that were not included in past problem sets or in past tutorials. You do not need to turn in any of these problems.

1. Read the explanation about the functions arcsin $x$ and arctan $x$ on pages 378–382 of the book (skip the references to integrals). In this question, you are going to do the same study for arccos $x$.

   (a) Sketch the graph of the function $y = \cos x$. What are its domain and its range? Prove that it is not one-to-one.

   (b) Consider the restriction of the function $\cos$ to the domain $[0, \pi]$. Now it is one-to-one. Let us call $\alpha$ the inverse function of this restriction of $\cos$. What are the domain and range of the function $\alpha$? Sketch its graph.

   (c) Consider the restriction of the function $\cos$ to the domain $[-\pi, 0]$. Now it is one-to-one. Let us call $\beta$ the inverse function of this restriction of $\cos$. What are the domain and range of the function $\beta$? Sketch its graph.

   (d) Write down the equations equivalent to (7.7.1) and (7.7.2) in the book, but using $\cos$ and $\alpha$ instead of $\sin$ and arcsin. Then write the equivalent equations using $\cos$ and $\beta$ instead of $\sin$ and arcsin. Make sure to indicate for which values of the variables they are true.

   (e) Calculate, if defined:

      i. $\alpha(\cos 2)$
      ii. $\alpha(\cos 4)$
      iii. $\alpha(\cos 8)$
      iv. $\cos(\alpha(5))$
      v. $\cos(\alpha(1/5))$

      i. $\beta(\cos 2)$
      ii. $\beta(\cos 4)$
      iii. $\beta(\cos 8)$
      iv. $\cos(\beta(5))$
      v. $\cos(\beta(1/5))$

   (f) Obtain a formula for the derivative of $\alpha$ imitating the calculation on top of page 380. Then obtain a formula for the derivative of $\beta$. 
2. Victor and Ivan are tied at opposite ends of a rope of fixed length. The rope is fully stretched; it starts at Ivan’s foot, then passes through a fixed point $P$ on the ceiling, and then continues to Victor’s foot. The rope is the dashed line in the picture below. The point $P$ is on the ceiling, 4m directly above the point $Q$ on the floor. Ivan is running away from Victor while Victor is just being dragged along the floor. How fast is Victor moving at the time when Ivan is 4m away from point $Q$, the distance between Victor and Ivan is 6m, and Ivan’s velocity is 1m/s?

3. In this problem, we will study the following two functions:

$$
\diamond(x) = \frac{e^x + e^{-x}}{2}, \quad \heartsuit(x) = \frac{e^x - e^{-x}}{2}.
$$

(a) Compute $\diamond'(x)$ and $\heartsuit'(x)$.

(b) Simplify the expression $\diamond^2(x) - \heartsuit^2(x)$.

*Note:* $\diamond^2(x)$ means $(\diamond(x))^2$.

(c) Use the equation you got for $\heartsuit'$ to prove that $\heartsuit$ must be one-to-one. Let ♠ be its inverse function.

(d) Find an explicit formula for $♠(y)$.

*Note:* If you are having trouble finding an expression for the inverse, consider the following easier questions first:

- Solve for $t$: $t^2 + 6t + 4 = 0$.
- Solve for $u$: $e^{2u} + 6e^u + 4 = 0$.
- Solve for $u$: $e^{2u} + 6ae^u + 4 = 0$.

(e) Use your answer to Question 3d to obtain a formula for $♠'(y)$. 
(f) There is a faster way to obtain a formula for ♠'(y) without having to obtain an explicit formula for ♠(y) first! Start with the identity

\[ \diamondsuit(♠(y)) = y, \]

take the derivative with respect to \( y \) on both sides, and use Questions 3b and 3a to obtain a formula for ♠'(y). This should agree with your result to Question 3e.

4. Let

\[ f(x) = \frac{(x^7 + 3x - 3)^{10} \sqrt{x^2 + 5x + 3}}{\sqrt{x^2 + 2x + 5}}. \]

Find the equation of the line tangent to the graph of \( f \) at the point with \( x \)-coordinate 1.

**Hint:** This is a short-ish computation if you use logarithmic differentiation.

5. Reread the formal definition of \( \lim_{x \to c} f(x) = L \) (this is Definition 2.2.1 on page 64 of the textbook). Then review Question 6 on Problem Set A. We now have new types of limit. For example, here is the definition of \( \lim_{x \to \infty} f(x) = L \):

Let \( f \) be a function defined at least on an open interval \((p, \infty)\) for some real number \( p \). We say that

\[ \lim_{x \to \infty} f(x) = L \]

when for every \( \varepsilon > 0 \), there exists \( M \in \mathbb{R} \) such that

if \( x > M \), then \( |f(x) - L| < \varepsilon \).

(a) Write a formal definition of the concept \( \lim_{x \to \infty} f(x) = L \).

(b) Write a formal definition of the concept \( \lim_{x \to \infty} f(x) = \infty \).

(c) Prove, from the formal definition, that \( \lim_{x \to \infty} \frac{x}{x + 1} = 1 \).