MAT 137Y: Calculus!

Problem Set 7.

Solutions

1. We know the following about the functions $\alpha$ and $\beta$:

$$\alpha'(x) = \beta(x), \quad \beta'(x) = \alpha(x).$$

Notes: Do not try to guess a formula for the functions $\alpha$ and $\beta$. There are many pairs of functions with these properties. Also, notice that $\alpha^n(x)$ means $(\alpha(x))^n$.

Calculate the following antiderivatives. Make sure to justify all your answers.

(a) $\int \beta(2x + 1) \, dx = \frac{1}{2} \alpha(2x + 1) + C$

Solution: Let $u = 2x + 1$, so that $du = 2 \, dx$ (or equivalently $dx = \frac{1}{2} \, du$). We have

$$\int \beta(2x + 1) \, dx = \int \frac{1}{2} \beta(u) \, du = \frac{1}{2} \alpha(u) + C = \frac{1}{2} \alpha(2x + 1) + C.$$

(b) $\int \frac{\alpha(x)}{\beta(x)} \, dx = \ln |\beta(x)| + C$

Solution: Let $u = \beta(x)$, so that $du = \alpha(x) \, dx$. We have

$$\int \frac{\alpha(x)}{\beta(x)} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\beta(x)| + C.$$

(c) $\int \alpha(2x)\beta^4(2x) \, dx = \frac{1}{10} \beta^5(2x) + C$

Solution: Let $u = \beta(2x)$, so that $du = 2\beta'(2x) \, dx = 2\alpha(2x) \, dx$ (or equivalently $\frac{1}{2} \, du = \alpha(2x) \, dx$). We have

$$\int \alpha(2x)\beta^4(2x) \, dx = \int \frac{1}{2} u^4 \, du = \frac{1}{10} u^5 + C = \frac{1}{10} \beta^5(2x) + C.$$

(d) $\int x^2 \alpha(x) \, dx = x^2 \beta(x) - 2x \alpha(x) + 2 \beta(x) + C$

Solution: Let $u = x^2$ and $dv = \alpha(x) \, dx$, so that $du = 2x \, dx$ and $v = \beta(x)$. Integrating by parts, we have

$$\int x^2 \alpha(x) \, dx = x^2 \beta(x) - 2 \int x \beta(x) \, dx. \quad (1)$$
In light of (1), let us compute \( \int x\beta(x)\,dx \) by parts. Set \( u = x \) and \( dv = \beta(x)\,dx \), so that \( du = dx \) and \( v = \alpha(x) \). We have

\[
\int x\beta(x)\,dx = x\alpha(x) - \int \alpha(x)\,dx = x\alpha(x) - \beta(x) + C. \tag{2}
\]

Combining (1) and (2), we find that

\[
\int x^2\alpha(x)\,dx = x^2\beta(x) - 2x\alpha(x) + 2\beta(x) + C.
\]

(e) \( \int \sin x\alpha(x)\,dx = \frac{1}{2}(\sin x\beta(x) - \cos x\alpha(x)) + C \)

**Solution:** Let \( u = \alpha(x) \) and \( dv = \sin x\,dx \), so that \( du = \beta(x)\,dx \) and \( v = -\cos x \). Integrating by parts, we have

\[
\int \sin x\alpha(x)\,dx = -\cos x\alpha(x) + \int \cos x\beta(x)\,dx. \tag{3}
\]

In light of (3), let us compute \( \int \cos x\beta(x)\,dx \) by parts. Set \( u = \beta(x) \) and \( dv = \cos x\,dx \), so that \( du = \alpha(x)\,dx \) and \( v = \sin x \). We have

\[
\int \cos x\beta(x)\,dx = \sin x\beta(x) - \int \sin x\alpha(x)\,dx.
\]

Substituting the above into (3), we obtain

\[
\int \sin x\alpha(x)\,dx = -\cos x\alpha(x) + \sin x\beta(x) - \int \sin x\alpha(x)\,dx.
\]

Rearranging this appropriately, we find that

\[
\int \sin x\alpha(x)\,dx = \frac{1}{2}(\sin x\beta(x) - \cos x\alpha(x)) + C.
\]

(f) \( \int x\sin x\alpha(x)\,dx = \frac{1}{2}(x\sin x\beta(x) - x\cos x\alpha(x) + \cos x\beta(x)) + C \)

**Solution:** Let \( u = x \) and \( dv = \sin x\alpha(x)\,dx \), so that \( du = dx \). Also, by Part (e), \( v = \frac{1}{2}(\sin x\beta(x) - \cos x\alpha(x)) \). Integrating by parts, we obtain

\[
\int x\sin x\alpha(x)\,dx = \frac{1}{2}x(\sin x\beta(x) - \cos x\alpha(x)) - \int \frac{1}{2} (\sin x\beta(x) - \cos x\alpha(x))\,dx
\]

\[
= \frac{1}{2}x(\sin x\beta(x) - \cos x\alpha(x)) - \frac{1}{2} \int \sin x\beta(x)\,dx + \frac{1}{2} \int \cos x\alpha(x)\,dx. \tag{4}
\]
In light of (4), we address ourselves to computing $\int \sin x \beta(x) \, dx$ and $\int \cos x \alpha(x) \, dx$. However, using a process analogous to that used in Part (e), one obtains

$$\int \sin x \beta(x) \, dx = \frac{1}{2}(\sin x \alpha(x) - \cos x \beta(x)) + C. \quad (5)$$

Similarly, we have

$$\int \cos x \alpha(x) \, dx = \frac{1}{2}(\sin x \alpha(x) + \cos x \beta(x)) + C. \quad (6)$$

Substituting (5) and (6) into (4), we obtain

$$\int x \sin x \alpha(x) \, dx = \frac{1}{2}x(\sin x \beta(x) - \cos x \alpha(x))$$

$$- \frac{1}{4}(\sin x \alpha(x) - \cos x \beta(x))$$

$$+ \frac{1}{4}(\sin x \alpha(x) + \cos x \beta(x))$$

$$+ C$$

$$= \frac{1}{2}(x \sin x \beta(x) - x \cos x \alpha(x) + \cos x \beta(x)) + C.$$

2. Continuing with the notation of Question 1, assume we also know that

$$\alpha^2(x) - \beta^2(x) = 1.$$ 

Calculate the following antiderivatives. Make sure to justify all your answers.

(a) $\int \alpha^5(x)\beta^4(x) \, dx = \frac{1}{9}\beta^9(x) + \frac{2}{7}\beta^7(x) + \frac{1}{5}\beta^5(x) + C.$

**Solution:** Note that

$$\alpha^5(x)\beta^4(x) = \alpha^4(x)\beta^4(x)\alpha(x)$$

$$= (\alpha^2(x))^2\beta^4(x)\alpha(x)$$

$$= (\beta^2(x) + 1)^2\beta^4(x)\alpha(x)$$

$$= (\beta^8(x) + 2\beta^6(x) + \beta^4(x))\alpha(x).$$

Hence,

$$\int \alpha^5(x)\beta^4(x) \, dx = \int (\beta^8(x) + 2\beta^6(x) + \beta^4(x))\alpha(x) \, dx.$$ 

Let $u = \beta(x)$, so that $du = \alpha(x) \, dx$. We have
\[
\int \alpha^5(x)\beta^4(x) \, dx = \int (u^8 + 2u^6 + u^4) \, du \\
= \frac{1}{9} u^9 + \frac{2}{7} u^7 + \frac{1}{5} u^5 + C \\
= \frac{1}{9} \beta^9(x) + \frac{2}{7} \beta^7(x) + \frac{1}{5} \beta^5(x) + C.
\]

(b) \[\int \frac{\beta^3(x)}{\alpha^4(x)} \, dx = -(\alpha(x))^{-1} + \frac{1}{3} (\alpha(x))^{-3} + C\]

**Solution:** Note that
\[
\frac{\beta^3(x)}{\alpha^4(x)} = \frac{\beta^2(x)}{\alpha^4(x)} \frac{\beta(x)}{\alpha(x)} \\
= \frac{\alpha^2(x) - 1}{\alpha^4(x)} \beta(x) \\
= ((\alpha(x))^{-2} - (\alpha(x))^{-4})\beta(x).
\]

Hence,
\[
\int \frac{\beta^3(x)}{\alpha^4(x)} \, dx = \int ((\alpha(x))^{-2} - (\alpha(x))^{-4})\beta(x) \, dx.
\]

Let \( u = \alpha(x) \), so that \( du = \beta(x) \, dx \). We have
\[
\int \frac{\beta^3(x)}{\alpha^4(x)} \, dx = \int (u^{-2} - u^{-4}) \, du \\
= -u^{-1} + \frac{1}{3} u^{-3} + C \\
= -(\alpha(x))^{-1} + \frac{1}{3} (\alpha(x))^{-3} + C.
\]

(c) \[\int \frac{\alpha^2(x)}{\beta^4(x)} \, dx = -\frac{\alpha^3(x)}{3\beta^3(x)} + C\]

**Hint:** For Question 2c, consider the function \( \gamma(x) = \frac{\alpha(x)}{\beta(x)} \) and compute \( \gamma'(x) \).

**Solution:** Noting the hint, we have
\[
\gamma'(x) = \frac{\beta(x)\alpha'(x) - \alpha(x)\beta'(x)}{\beta^2(x)} = \frac{\beta^2(x) - \alpha^2(x)}{\beta^2(x)} = -\frac{1}{\beta^2(x)}.
\]
Accordingly, we obtain
\[ \int \frac{\alpha^2(x)}{\beta^4(x)} \, dx = \int \left( \frac{\alpha(x)}{\beta(x)} \right)^2 \left( \frac{1}{\beta^2(x)} \right) \, dx = -\int \gamma^2(x) \gamma'(x) \, dx. \]

Let \( u = \gamma(x) \), so that \( du = \gamma'(x) \, dx \). Our integral then becomes
\[
\int \frac{\alpha^2(x)}{\beta^4(x)} \, dx = -\int u^2 \, du
= -\frac{1}{3} u^3 + C
= -\frac{\alpha^3(x)}{3\beta^3(x)} + C.
\]
3. A hemispherical salad bowl has radius $R$. We place an iron ball with radius $r$ inside the bowl. Assume that $2r \leq R$. Then we pour water into the bowl until the height of water is $h$.

(a) Assume $0 \leq h \leq 2r$. Calculate the total volume of water.

**Solution:** Consider the circle of radius $R$ centred at $(0, R)$, and let $C_R$ denote the part of this circle for which $x \in [0, R]$ and $y \in [0, R]$. Noting that the circle itself has equation $x^2 + (y - R)^2 = R^2$, one finds that $C_R$ is the curve

$$x = \sqrt{R^2 - (y - R)^2}, \quad y \in [0, R].$$

Secondly, consider the circle of radius $r$ centred at $(0, r)$, and let $C_r$ denote the part lying in the first quadrant. Since the circle itself has equation $x^2 + (y - r)^2 = r^2$, one notes that $C_r$ is the curve

$$x = \sqrt{r^2 - (y - r)^2}, \quad y \in [0, 2r].$$

Note that $C_R$, $C_r$, and the line $y = R$ bound a region $\Omega$, as depicted below.

![Diagram](image)

By rotating $\Omega$ around the $y$-axis, one obtains the solid $S$ between the iron ball and the salad bowl. For $y \in [0, R]$, let $A(y)$ denote the area of the cross-section of $S$ at $y$. If the height of the water is $h$, then the total volume $V(h)$ of water is

$$V(h) = \int_0^h A(y) \, dy.$$ (7)
Since \(0 \leq h \leq 2r\), we need only find \(A(y)\) when \(0 \leq y \leq 2r\). In this case,

\[
A(y) = \pi \left( \sqrt{R^2 - (y - R)^2} \right)^2 - \pi \left( \sqrt{r^2 - (y - r)^2} \right)^2
= \pi (R^2 - (y^2 - 2Ry + R^2) - r^2 + (y^2 - 2ry + r^2))
= 2\pi (R - r)y.
\]

Using (7), it follows that

\[
V(h) = \int_0^h 2\pi (R - r)y \, dy
= \left[ \pi (R - r)y^2 \right]^h_0
= \pi (R - r)h^2.
\]

(b) Assume \(2r \leq h \leq R\). Calculate the total volume of water.

**Solution:** Note that (7) can be written as

\[
V(h) = \int_0^{2r} A(y) \, dy + \int_{2r}^h A(y) \, dy = V(2r) + \int_{2r}^h A(y) \, dy.
\]

Also, (8) yields \(V(2r) = \pi (R - r) (2r)^2 = 4\pi (Rr^2 - r^3)\), so that

\[
V(h) = 4\pi (Rr^2 - r^3) + \int_{2r}^h A(y) \, dy.
\]

Furthermore, when \(2r \leq y \leq R\),

\[
A(y) = \pi \left( \sqrt{R^2 - (y - R)^2} \right)^2 = \pi (2Ry - y^2).
\]

Hence,

\[
V(h) = 4\pi (Rr^2 - r^3) + \int_{2r}^h \pi (2Ry - y^2) \, dy
= 4\pi (Rr^2 - r^3) + \left[ \pi Ry^2 - \frac{\pi}{3} y^3 \right]_{2r}^h
= 4\pi (Rr^2 - r^3) + \left( \pi Rh^2 - \frac{\pi}{3} h^3 - 4\pi Rr^2 + \frac{8\pi}{3} r^3 \right)
= \pi \left( Rh^2 - \frac{4}{3} r^3 - \frac{1}{3} h^3 \right).
\]
Checking the solutions: There are two checks we can perform:

- Part (a) computes the volumes when $0 \leq h \leq 2r$. Part (b) computes the volume when $2r \leq h \leq R$. When $h = 2r$, both solutions should agree. Yes, this is the case. When $h = 2r$ Equation (8) becomes:

$$V(2r) = \pi (R - r)(2r)^2 = 4\pi (R - r)r^2.$$  

On the other hand, when $h = 2r$ Equation (9) becomes:

$$V(2r) = \pi \left( R(2r)^2 - \frac{4}{3}r^3 - \frac{1}{3}(2r)^3 \right) = \pi (4Rr^2 - 4r^3).$$

They are equal, as they should be.

- When $h = R$, the volume of water equals the volume of half a sphere with radius $R$ minus the volume of a full sphere with radius $r$. Indeed, when $h = R$, Equation (9) becomes:

$$V(R) = \pi \left( RR^2 - \frac{4}{3}r^3 - \frac{1}{3}R^3 \right) = \frac{1}{2} \left[ \frac{4}{3} \pi R^3 \right] - \frac{4}{3} \pi r^3.$$
4. Two long cylinders have radii \( r \) and \( R \) respectively (with \( r \leq R \)) and their axes meet at a right angle. We want to compute the volume \( V \) of their intersection.

(a) Set up an integral whose value equals \( V \). Leave your answer indicated as an integral.

**Solution:** In the interest of helping the reader visualize what follows, suppose that the \( x \) and \( y \)-axes lie on the blackboard. We may consider a third axis, the \( z \)-axis, perpendicular to the \( x \)-\( y \) plane and coming out of the blackboard. Now, let \( C_R \) and \( C_r \) denote the cylinders of radii \( R \) and \( r \), respectively. For convenience, we will assume that the \( x \)-axis is the axis of \( C_R \), while the \( y \)-axis is the axis of \( C_r \). With this in mind, a point \((x, y, z)\) is on \( C_R \) if and only if
\[
y^2 + z^2 = R^2, \ x \in \mathbb{R}, \tag{10}
\]
while the point lies on \( C_r \) if and only if
\[
x^2 + z^2 = r^2, \ y \in \mathbb{R}, \tag{11}
\]
Our cylinders intersect in a solid whose volume we will compute by considering \( z \)-cross-sections. Each such cross-section is a rectangle with one side parallel to the \( x \)-axis and another side parallel to the \( y \)-axis. Let \( L(z) \) and \( \ell(z) \) denote the respective lengths of these sides. By recalling (10) and (11), noting that the corners of the rectangle lie on the cylinders, and drawing a suitable diagram, one concludes that \( (\frac{1}{2}L(z))^2 + z^2 = R^2 \) and \( (\frac{1}{2}\ell(z))^2 + z^2 = r^2 \). In other words,
\[
\ell(z) = 2\sqrt{R^2 - z^2}
\]
and
\[
L(z) = 2\sqrt{r^2 - z^2}.
\]
The area of our rectangular cross-section is therefore
\[
A(z) = L(z)\ell(z) = 4\sqrt{R^2 - z^2}\sqrt{r^2 - z^2},
\]
giving us
\[
V = \int_{-r}^{r} A(z) \, dz = 4 \int_{-r}^{r} \sqrt{R^2 - z^2}\sqrt{r^2 - z^2} \, dz. \tag{12}
\]
(b) Let \( a = R/r \). Use a substitution to show that \( V = r^3F(a) \). In other words, we can write the volume \( V \) as \( r^3 \) times an expression that depends only on \( a \) (and not on \( R \) or \( r \) separately). Write a formula for \( F \).
Solution: First, let us rewrite (12) as

\[ V = 4 \int_{-r}^{r} \sqrt{R^2 - z^2} \sqrt{r^2 - z^2} \, dz = 4 \int_{-r}^{r} \sqrt{a^2 r^2 - z^2} \sqrt{r^2 - z^2} \, dz \]

\[ = 4 \int_{-r}^{r} r \sqrt{a^2 - \left( \frac{z}{r} \right)^2} r \sqrt{1 - \left( \frac{z}{r} \right)^2} \, dz. \]

This suggest using the substitution \( u = \frac{z}{r} \). Then \( du = \frac{1}{r} \, dz \) or \( dz = rdu \) and we get

\[ V = 4r^3 \int_{-1}^{1} \sqrt{a^2 - u^2} \sqrt{1 - u^2} \, du. \]

Hence, we have

\[ F(a) = 4 \int_{-1}^{1} \sqrt{a^2 - u^2} \sqrt{1 - u^2} \, du = 8 \int_{0}^{1} \sqrt{a^2 - u^2} \sqrt{1 - u^2} \, du. \]

Alternatively, we could have used the substitution \( z = (ar)t \), so that \( dz = (ar) \, dt \), and we would have gotten:

\[ F(a) = 8a^2 \int_{0}^{\frac{1}{a}} \sqrt{1 - t^2} \sqrt{1 - a^2 t^2} \, dt. \]

There are other, equivalent, correct integral expressions.