MAT 137Y: Calculus!
Problem Set 6.
Due in tutorial on February 1–2

Instructions:

• Print this cover page, fill it out entirely, sign at the bottom, and STAPLE it to the front of your problem set solutions. (You do not need to print the questions.)
  Doing this correctly is worth 1 mark.
• Submit your problem set ONLY in the tutorial in which you are enrolled.
• Before you attempt this problem set read the notes we posted online about the definition of the integral and do all the practice problems from section 12.1, 5.2, 5.3 (see course website).

PLEASE NOTE that so far over 20 students have been penalized for academic misconduct and now have a record with OSAI. Do not be the next one. Re-read “Important notes on collaboration” on the cover page for Problem Set 1.

Last name .................................................................
First name .................................................................
Student number ............................................................
Tutorial code ...............................................................
TA name .................................................................

Please, double-check your tutorial code on blackboard, and double-check your TA name on the course website. Remember that if there is a discrepancy between Blackboard and ROSI/ACORN, then your correct tutorial is the one on Blackboard, not on ROSI/ACORN. See http://uoft.me/137tutorials
0. Let $f$ be a bounded function on the real interval $[a, b]$.
   
   (a) Prove that $I^b_a(f)$ satisfies the following two properties:
   
   i. $L_f(P) \leq I^b_a(f)$ for every partition $P$ of $[a, b]$.
   
   ii. For every $\varepsilon > 0$, there exists a partition $P$ of $[a, b]$ such that $I^b_a(f) - \varepsilon < L_f(P) \leq I^b_a(f)$.
   
   (b) Let $J$ be a real number. Assume $J$ satisfies the following two properties:
   
   i. $L_f(P) \leq J$ for every partition $P$ of $[a, b]$.
   
   ii. For every $\varepsilon > 0$, there exists a partition $P$ of $[a, b]$ such that $J - \varepsilon < L_f(P) \leq J$.
   
   Prove that $J = I^b_a(f)$
   
   Note: This problem is very short once you are very comfortable with the definition of supremum and of lower integral.

1. Write the statement (without proof) of the results equivalent to Question 0 for $I^b_a(f)$ instead of $I^b_a(f)$.

2. Let $f$ be a bounded function on the interval $[a, b]$.
   
   (a) Assume that $f$ satisfies the following property:
   
   $\forall \varepsilon > 0, \exists$ partition $P$ of $[a, b]$, such that $U_f(P) - L_f(P) < \varepsilon$.
   
   Prove that $f$ is integrable on $[a, b]$.
   
   (b) Assume that $f$ is integrable on $[a, b]$. Prove that $f$ satisfies the following property:
   
   $\forall \varepsilon > 0, \exists$ partition $P$ of $[a, b]$, such that $U_f(P) - L_f(P) < \varepsilon$.
   
   Hint: Use the definition of integrability: $f$ is integrable on $[a, b]$ if and only if $T^b_a(f) - L^b_a(f) = 0$. Also use the definitions of $T^b_a(f)$ and $L^b_a(f)$. Finally, remember that we always know that $T^b_a(f) - L^b_a(f) \geq 0$, whether $f$ is integrable or not.

3. Let $f$ and $g$ be two bounded functions on the interval $[a, b]$.
   
   (a) Let $P$ be a partition of $[a, b]$. Only one of the following two inequalities is always true:
   
   $L_{f+g}(P) \leq L_f(P) + L_g(P), \quad L_{f+g}(P) \geq L_f(P) + L_g(P)$
   
   Determine which one is always true, prove it, and then show the other one is not always true with an example.
(b) Repeat Question 3a with upper sums instead of lower sums.

(c) Assume that $f$ and $g$ are integrable on $[a, b]$. Prove that $f + g$ is also integrable on $[a, b]$.
Hint: Use Problem 2 repeatedly.

4. Give an example of two bounded functions $f$ and $g$ on an interval $[a, b]$ such that $\int_a^b (f + g) \neq \int_a^b f + \int_a^b g$.

5. In this question, you are going to compute the exact value of $\int_2^5 (5x - x^2) \, dx$ using Riemann sums. Let us call $f(x) = 5x - x^2$. Since $f$ is continuous on $[1, 3]$, we know it is integrable. Hence, its value can be computed using any Riemann sums via equation (5.2.7) in the book. For every natural number $n$, let us call $P_n$ the partition that splits $[2, 5]$ into $n$ equal sub-intervals. Notice that $\lim_{n \to \infty} ||P_n|| = 0$. Hence, we can write

$$\int_2^5 (5x - x^2) \, dx = \lim_{n \to \infty} S(P_n),$$

where $S(P_n)$ is any Riemann sum for $f$ and $P_n$. In particular, to make things simpler, we are going to choose the Riemann sum $S(P_n)$ where at every subinterval we use the right-endpoint to evaluate $f$.

(a) Let us write $P_n = \{x_0, x_1, \ldots, x_n\}$. Find a formula for $x_i$ in terms of $i$ and $n$.

(b) What is the length of each sub-interval in $P_n$?

(c) Since we are using the right-endpoint, it means we are picking $x_i^* = x_i$. Use your above answers to obtain an expression for $S(P_n)$ in the form of a sum with sigma notation.

(d) Using the formulas

$$\sum_{i=1}^N i = \frac{N(N + 1)}{2}, \quad \sum_{i=1}^N i^2 = \frac{N(N + 1)(2N + 1)}{6}, \quad \sum_{i=1}^N i^3 = \frac{N^2(N + 1)^2}{4}$$

if needed, add up the expression you got to obtain a nice, compact formula for $S(P_n)$ without any sums or sigma symbols.

(e) Calculate $\lim_{n \to \infty} S(P_n)$. This number will be the exact value of $\int_2^5 (5x - x^2) \, dx$.
Hint: Your final answer should be $\frac{27}{2}$. 

4.