1. A cop is trying to catch drivers who speed on the highway. She finds a long stretch of the highway. She parks her car behind some bushes, 400 metres away from the highway. There is a traffic sign at the point of the road closest to her car, and there is a phone by the road 600 metres away from the traffic sign.

(a) The cop points her radar gun at a car and learns that, as the car is passing by the phone, the distance between the car and the cop is increasing at a rate of 80 km/h. The speed limit is 120 km/h. Can she fine the driver?

Solution:

Remark. Notice that this question measures distances in both metres and km. In the interest of determining the speed of the car in km/h, our solution regards all distances as being measured in km.

The driver is moving in a straight line (the only non-dashed line in the picture below). Let us call $x$ the distance from the sign to the driver and $y$ the distance from the cop to the driver. We want to compute the speed of the driver, which is $\frac{dx}{dt}$ at the time when $\frac{dy}{dt} = 80 \text{ km/h}$ and $x = 0.6 \text{ km}$. Finally, notice that the distance from the sign to the cop is constant and equal to $a = 0.4 \text{ km}$.

We know that

\[ y^2 = x^2 + a^2 \]  

(1)

Differentiating implicitly with respect to time, we get

\[ 2y \frac{dy}{dt} = 2x \frac{dx}{dt} \]  

(2)

At the time in question, from (1), we have that $y = \sqrt{0.6^2 + 0.4^2} = \sqrt{0.52} \text{ km}$. Now we simply substitute into Equation (2):

\[
\frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt} = \frac{\sqrt{0.52}}{0.6} \frac{80}{3} \approx 96.15 \text{ km/h}
\]
As this is beneath the speed limit, the cop cannot fine the driver.

(b) Why doesn’t the cop point her radar gun at cars as they pass by the traffic sign rather than as they pass by the phone?

Solution: Suppose that the cop had decided to point her radar gun at the traffic sign. Also, let \( x \) and \( y \) be exactly as defined in the solution to part (a). Both Equations (1) and (2) are still true. The cop is still trying to obtain the speed of the car, which is \( \frac{dx}{dt} \). In this case, however, the cop is trying to determine it when \( x = 0 \). Hence, she needs to work with the equation:

\[
0 \cdot \frac{dx}{dt} = 2y \frac{dy}{dt}. \tag{3}
\]

Notice that \textbf{the rather gun will always produce the measurement} \( \frac{dy}{dt} = 0 \) in this situation. In other words, regardless of the speed of the car, the cop will always record the same measurement. Hence, she will not be able to determine the speed of the car.

There is an alternative, algebraic explanation. We may view solving (3) for \( \frac{dx}{dt} \) as solving \( 0 \cdot a = 0 \) for \( a \). Note, however, that there isn’t a unique solution for \( a \) in this equation.

\textit{Warning:} The fact that \( \frac{dy}{dt} = 0 \) does \textit{not} mean that \( y \) is constant. It only means that the instantaneous rate of change of \( y \) at that point is zero. Indeed, \( y \) is the distance between the cop and the driver. At that point, \( y \) reaches a minimum, and hence its derivative is 0.

2. The equation \( x^5y + x + y^3 = 3 \) defines implicitly a function \( y = g(x) \) near \( x = 1 \). Compute \( g(1) \), \( g'(1) \), and \( g''(1) \).

Solution: We begin with an important remark.

\textit{Remark.} One approach is to find a formula (written in terms of \( x \) and \( y \)) for each of \( g'(x) = \frac{dy}{dx} \) and \( g''(x) = \frac{d^2y}{dx^2} \). While such an approach is \textbf{absolutely fine} and can certainly be made to work, note that this question does not ask for \( g'(x) \) and \( g''(x) \). Rather, it asks for \( g'(1) \) and \( g''(1) \) (as well as \( g(1) \)). The following solution is therefore designed to emphasize how one can compute \( g'(1) \) and \( g''(1) \) without finding expressions for \( g'(x) \) and \( g''(x) \). We believe our approach to be the faster of the two.

Before we begin, let us interpret the situation described in the statement of the question. Suppose that \( x \) a fixed number and that \( x \) is near 1. In this case, one can
think of solving
\[ x^5 y + x + y^3 = 3 \]  
(4)
for the one remaining unknown quantity \( y \). Furthermore, the unique solution \( y \) is precisely \( g(x) \).

In particular, \( y = g(1) \) is the unique solution of (4) when \( x = 1 \). In other words, \( y = g(1) \) is the unique solution of
\[ y + 1 + y^3 = 3. \]  
(5)
By inspection, \( y = 1 \) solves (5). It follows that \( g(1) = 1 \).

In order to compute \( g'(1) = \frac{dy}{dx} \bigg|_{x=1} \), we will need some information about \( g'(x) = \frac{dy}{dx} \). We will obtain this information via (4) and implicit differentiation. For simplicity, let us write \( y' \) instead of \( \frac{dy}{dx} \). Differentiating (4) with respect to \( x \), we get
\[ \frac{d}{dx} (x^5 y + x + y^3) = 0 \]
\[ 5x^4 y + x^5 y' + 1 + 3y^2 y' = 0 \]  
(6)
Since \( g(1) = 1 \), it follows that \( y = 1 \) when \( x = 1 \). We may therefore substitute \( x = 1 \) and \( y = 1 \) into (6) to get
\[ 5 + y' + 1 + 3y' = 0 \]
From this we obtain
\[ g'(1) = y' \bigg|_{x=1} = \frac{-3}{2}. \]
Next, in order to compute \( g''(1) = y'' \bigg|_{x=1} \), we differentiate Equation (6):
\[ \frac{d}{dx} \left[ 5x^4 y + x^5 y' + 1 + 3y^2 y' \right] = 0 \]
\[ 20x^3 y + 5x^4 y' + 5x^4 y' + x^5 y'' + 6yy'y' + 3y^2 y'' = 0 \]  
(7)
Then we substitute \( x = 1 \), \( y = 1 \), and \( y' = -3/2 \) into Equation (6):
\[ 20 + 5 \frac{-3}{2} + 5 \frac{-3}{2} + y'' + 6 \left( \frac{-3}{2} \right)^2 + 3y'' = 0 \]
From this we obtain:
\[ g''(1) = y'' \bigg|_{x=1} = \frac{-37}{8}. \]
3. Read Equation 7.1.7 on page 338 of the book and its derivation. Another way to write the same equation is the following. If \( f \) is a differentiable, one-to-one function and \( f(a) = b \), then

\[
(f^{-1})'(b) = \frac{1}{f'(a)}
\]

Obtain (with proof) similar equations for \((f^{-1})''(b)\) and for \((f^{-1})'''(b)\) in terms of \( f'(a), f''(a), \) and \( f'''(a) \).

**Solution:**

**Remark.** In the interest of being able to divide by \( f'(a) \), the following solution will assume that \( f'(a) \neq 0 \). Indeed, note that this assumption is necessary when one is proving that \((f^{-1})'(b) = \frac{1}{f'(a)}\).

As in the derivation of Equation 7.1.7, we begin by noting that \( f^{-1}(f(x)) = x \). Hence,

\[
\frac{d}{dx} [f^{-1}(f(x))] = 1
\]

\[
(f^{-1})'(f(x)) f'(x) = 1. \tag{8}
\]

While one could use (8) to get an expression for \((f^{-1})'(b)\), the exercise is to instead obtain expressions for \((f^{-1})''(b)\) and \((f^{-1})'''(b)\). With this in mind, let us differentiate both sides of (8).

\[
\frac{d}{dx} [(f^{-1})'(f(x)) f'(x)] = 0
\]

\[
(f^{-1})'(f(x)) \frac{d}{dx} [f'(x)] + f'(x) \frac{d}{dx} [(f^{-1})'(f(x))] = 0
\]

\[
(f^{-1})'(f(x)) f''(x) + (f^{-1})''(f(x))(f'(x))^2 = 0. \tag{9}
\]

Evaluating (9) when \( x = a \), we obtain

\[
(f^{-1})'(b) f''(a) + (f^{-1})''(b)(f'(a))^2 = 0.
\]

It follows that

\[
(f^{-1})''(b) = -\frac{(f^{-1})'(b) f''(a)}{(f'(a))^2}
\]

\[
= -\left(\frac{1}{f'(a)}\right) f''(a)
\]

\[
= -\frac{f''(a)}{(f'(a))^3}.
\]
Finally, seeking a third derivative, we differentiate both sides of (9).

\[
\frac{d}{dx} \left[ (f^{-1})'(f(x))f''(x) + (f^{-1})''(f(x))(f'(x))^2 \right] = 0
\]

\[
(f^{-1})'(f(x)) \frac{d}{dx} [f''(x)] + f''(x) \frac{d}{dx} [(f^{-1})'(f(x))]
\]
\[
+ (f^{-1})''(f(x)) \frac{d}{dx} [(f'(x))^2] + (f'(x))^2 \frac{d}{dx} [(f^{-1})''(f(x))] = 0
\]

\[
(f^{-1})'(f(x))f''(x) + f'(x)f''(x)(f^{-1})''(f(x))
\]
\[
+ 2f'(x)f''(x)(f^{-1})''(f(x)) + (f'(x))^3(f^{-1})'''(f(x)) = 0
\]

(10)

Evaluating (10) when \(x = a\), we obtain

\[
(f^{-1})'(b)f''(a) + 3f'(a)f''(a)(f^{-1})''(b) + (f'(a))^3(f^{-1})'''(b) = 0.
\]

Solving the above for \((f^{-1})'''(b)\) yields

\[
(f^{-1})'''(b) = \frac{-(f^{-1})'(b)f''(a) + 3f'(a)f''(a)(f^{-1})''(b)}{(f'(a))^3}
\]
\[
= -\frac{\left(\frac{1}{f'(a)}\right)f'''(a) + 3f'(a)f''(a)\left(-\frac{f''(a)}{(f'(a))^2}\right)}{(f'(a))^3}
\]
\[
= -\frac{f'(a)f'''(a) - 3(f''(a))^2}{(f'(a))^5}
\]
\[
= \frac{3(f''(a))^2 - f'(a)f'''(a)}{(f'(a))^5}.
\]
4. Two ants are taking a nap. The first one is resting at the tip of the minute hand of a cuckoo clock, which is 25 cm long. The second one is resting at the tip of the hour hand, which is half the length. At what rate is the distance between the two ants changing at 3:30?

**Solution:** Let $\theta$ denote the angle between the minute and hour hands, and let $y$ denote the distance between the ants. Let $a = 25\text{ cm}$ and $b = 12.5\text{ cm}$ be the lengths of the hands. Notice that $a$ and $b$ are constants, whereas $\theta$ and $y$ are functions of time.

Our objective is to compute $\frac{dy}{dt}$ at 3:30. We begin by finding a relation between $y$ and $\theta$. The Law of Cosines implies that

$$y^2 = a^2 + b^2 - 2ab \cos \theta \quad (11)$$

**Note:** It is entirely possible to solve this problem without using the Law of Cosines. See an alternative solution at the end.

Differentiating Equation (11) with respect to time we get:

$$2y \frac{dy}{dt} = 2ab \sin \theta \frac{d\theta}{dt} \quad (12)$$
Notice that we can compute $\frac{dy}{dt}$ from Equation (12) if we are able to compute $a$, $b$, $y$, $\theta$, and $\frac{d\theta}{dt}$ at 3:30. We will do that next.

- We know that $a = 25\,cm$ and $b = 12.5\,cm$.
- At 3:30, the tip of hour hand (if extended to the edge of the clock) is halfway between the positions of 3 and 4, while the tip of the minute hand is at the position of 6. In other words, the two hands are separated by 2.5 of the 12 hours on the clock. Hence,

$$\theta = \frac{2.5}{12} \cdot 2\pi = \frac{5\pi}{12}$$

- Next, from Equation (11), we can evaluate

$$y = \sqrt{a^2 + b^2 - 2ab \cos \theta} = \sqrt{781.25 - 625 \cos \frac{5\pi}{12}} \approx 24.9\,cm.$$  

- Finally, let us compute $\frac{d\theta}{dt}$ at 3:30. Actually, the quantity $\frac{d\theta}{dt}$ is constant. Note that the minute hand cycles through $2\pi$ radians every hour, whereas the hour hand cycles goes through $\frac{2\pi}{12}$ radians every hour. Hence, the angle between both hands changes at a rate of

$$\frac{d\theta}{dt} = 2\pi - \frac{2\pi}{12} = \frac{11\pi}{6} \text{ rad/h}$$

**Warning:** We absolutely must measure $\frac{d\theta}{dt}$ in radians per hour (or radians per minute or radians per second...) rather than in degrees per hour (or degrees per minute or degrees per second...) Why is this? In our derivation, we used the following fact: $\frac{d}{dt} \cos \theta = -\sin \theta \frac{d\theta}{dt}$. In other words, we used the fact that $\frac{d}{d\theta} \cos \theta = -\sin \theta$. This identity, like most identities involving derivatives or limits of trig functions, is only true when $\theta$ is measured in radians, not in degrees.

Plugging all of the values into Equation (12) we obtain

$$\frac{dy}{dt} = \frac{ab \sin \theta}{y} \frac{d\theta}{dt} = \frac{25 \cdot 12 \cdot \sin \frac{5\pi}{12}}{\sqrt{781.25 - 625 \cos \frac{5\pi}{12}}} \cdot \frac{11\pi}{6} \text{ cm/h}$$

$$\approx 69.9 \text{ cm/h} \approx 1.16 \text{ cm/min} \approx 0.194 \text{ mm/s}$$
In other words, the distance between the two ants is increasing at an approximate rate of 1.16 cm/min. This concludes the solution.

Alternative Derivation of (11) without knowing the Law of Cosines

Let $\alpha$ be the angle from the vertical axis (12 o’clock) and the minute hand, measured clockwise. Let $\beta$ be the angle from the vertical axis (12 o’clock) and the hour hand, measured clockwise. Notice that $\theta = \alpha - \beta$. For example, at 3:30 we have $\alpha = \pi$ and $\beta = \frac{7\pi}{12}$. Remember that $a$ and $b$ are the lengths of the minute and the hour hand, respectively.

At any time, the position of the tip of the minute hand will be $(a \sin \alpha, a \cos \alpha)$. The position of the hour hand will be $(b \sin \beta, b \cos \beta)$. The distance between those two points is $y$, and it satisfies:

$$y^2 = (a \sin \alpha - b \sin \beta)^2 + (a \cos \alpha - b \cos \beta)^2$$
$$= a^2(\sin^2 \alpha + \cos^2 \alpha) + b^2(\sin^2 \beta + \cos^2 \beta) - 2ab(\sin \alpha \sin \beta + \cos \alpha \cos \beta)$$
$$= a^2 + b^2 - 2ab \cos(\alpha - \beta)$$
$$= a^2 + b^2 - 2ab \cos \theta.$$ 

This is exactly the same as Equation (11).