1. Here is the graph of a function $f$:

![Graph of function $f$]

Find the following limits for this function:

(a) $\lim_{x \to 2^+} f(x)$
(b) $\lim_{x \to 2^-} f(x)$
(c) $\lim_{x \to 2} f(x)$
(d) $\lim_{x \to 0^+} f(f(x))$
(e) $\lim_{x \to 0^+} f(f(x))$
(f) $\lim_{x \to 2} f(f(2))$
(g) $\lim_{x \to 0} [f(x)]^2$
(h) $\lim_{x \to 0} f(2\cos x)$
(i) $\lim_{x \to 0} f(2\sec x)$

Solution:

(a) $\lim_{x \to 2^+} f(x) = 2$. This follows from an examination of the graph.

(b) $\lim_{x \to 2^-} f(x) = -2$. This follows from an examination of the graph.

(c) $\lim_{x \to 2} f(x)$ does not exist. This follows from the fact that the one-sided limits $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$ do not agree, as established in (a) and (b).

To answer questions (d)-(f), let us look closely at the composition $y = f(f(x))$. For ease of notation, I am going to call $t = f(x)$ and $y = f(f(x)) = f(t)$. 
First, notice that as \( x \to 0 \), we have that \( t = f(x) \to 2 \). But there is more: when \( x \) is close to 0, it does not matter whether \( x > 0 \) or \( x < 0 \); in both cases \( f(x) < 2 \). Hence we can write that

(d) \[
\lim_{x \to 0^+} f(f(x)) = \lim_{t \to 2^+} f(t) = -2.
\]

(e) \[
\lim_{x \to 0} f(f(x)) = \lim_{t \to 2^-} f(t) = -2.
\]

On the other hand, when \( x \) is close to \(-3\), then \( f(x) = 2 \) identically, and \( f(f(x)) = -1 \) identically. I'll repeat: when \( x \) is close to \(-3\), but not \(-3\), \( f(x) \) is already 2, and \( f(f(x)) \) is already \(-1\). Hence

(f) \[
\lim_{x \to -3} f(f(x)) = \lim_{x \to -3} f(2) = -1.
\]

Alternatively, in order to answer parts (d)-(f), we can first constructing the graph of \( y = f(f(x)) \). We sketch it below.

Then, the three questions can be answered by direct examination of the graph:

To answer part (g) we could sketch the graph of \( y = [f(x)]^2 \). Alternatively, we can notice that

\[
\lim_{x \to 2^+} [f(x)]^2 = (2)^2 = 4, \quad \lim_{x \to 2^-} [f(x)]^2 = (-2)^2 = 4.
\]

Since they are both equal, we obtain that:
(g) \( \lim_{x \to 2} [f(x)]^2 = 4. \)

To address parts (h) and (i), notice the following. As \( x \to 0 \), we have that \( \cos x \to 1^- \). In other words, when \( x \) is close to 0 (but not 0), \( \cos x < 1 \). Similarly, when \( x \to 0 \), \( \sec x > 1 \). This suggests that

(h) \( \lim_{x \to 0} f(2 \cos x) = \lim_{t \to 2^-} f(t) = -2. \)

(i) \( \lim_{x \to 0} f(2 \sec x) = \lim_{t \to 2^+} f(t) = 2. \)

2. In each one of the following cases, we want to find a pair of functions \( f \) and \( g \) with some properties. Decide for each case whether it is possible or not. If you answer YES, give us an example. if you answer NO, prove it.

(a) \( \lim_{x \to 0} f(x) \) does not exist, and \( \lim_{x \to 0} [f(x) + g(x)] = 42 \)

Solution: YES. Consider the functions \( f(x) = \frac{1}{x} \) and \( g(x) = -\frac{1}{x} + 42 \). It is true that \( \lim_{x \to 0} f(x) \) does not exist, while

\[
\lim_{x \to 0} \left( \frac{1}{x} + (-\frac{1}{x} + 42) \right) = \lim_{x \to 0} 42 = 42.
\]

(b) \( \lim_{x \to 0} f(x) = 0 \), and \( \lim_{x \to 0} [f(x)g(x)] = 42 \)

Solution: YES. Consider the functions \( f(x) = x \) and \( g(x) = \frac{42}{x} \). It is true that \( \lim_{x \to 0} f(x) = 0 \), while

\[
\lim_{x \to 0} \left( \frac{42}{x} \right) = \lim_{x \to 0} 42 = 42.
\]

(c) \( \lim_{x \to 0} f(x) = 0 \), and \( \lim_{x \to 0} [f(x)g(x)] = \infty \)

Solution: YES. Consider the functions \( f(x) = x \) and \( g(x) = \frac{1}{x^2} \). It is true that \( \lim_{x \to 0} f(x) = 0 \), while

\[
\lim_{x \to 0} \left( x \cdot \frac{1}{x^3} \right) = \lim_{x \to 0} \frac{1}{x^2} = \infty.
\]
(d) \( \lim_{x \to 1} f(x) = 2 \), \( \lim_{u \to 2} g(u) = 3 \), and \( \lim_{x \to 1} g(f(x)) = 42 \).

Solution: YES. Consider the functions \( f(x) = 2 \) and 
\[
g(x) = \begin{cases} 
3 & \text{if } x \neq 2 \\
42 & \text{if } x = 2 
\end{cases}
\]
We have that \( \lim_{x \to 1} f(x) = 2 \), \( \lim_{u \to 2} g(u) = 3 \), and
\[
\lim_{x \to 1} g(f(x)) = \lim_{x \to 1} g(2) = \lim_{x \to 1} 42 = 42.
\]

Remark. The following remark need not be included in a complete solution. Instead, it is intended to provide some context for the solution given above.

Informally speaking, we are told that as \( x \) approaches 1, \( f(x) \) approaches 2. Secondly, the condition \( \lim_{u \to 2} g(u) = 3 \) indicates that the output of \( g \) approaches 3 as its input approaches 2. Combining these two ideas, we might suspect that \( g(f(x)) \) approaches 3 as \( x \) approaches 1. We might therefore conjecture that \( \lim_{x \to 1} g(f(x)) = 3 \). Let us attempt to prove this, and then examine why our proof cannot be made to work.

We claim that \( \lim_{x \to 1} g(f(x)) = 3 \). Accordingly, let \( \varepsilon > 0 \) be given. Since \( \lim_{u \to 2} g(u) = 3 \), there exists \( \delta' > 0 \) such that
\[
0 < |u - 2| < \delta' \implies |g(u) - 3| < \varepsilon.
\]
Also, because \( \lim_{x \to 1} f(x) = 2 \), there exists \( \delta > 0 \) such that
\[
0 < |x - 1| < \delta \implies |f(x) - 2| < \delta'.
\]
We might try to combine (1) and (2) in the following way: If \( 0 < |x - 1| < \delta \), then \( |f(x) - 2| < \delta' \) (by (2)), so that we can apply (1) with \( u = f(x) \) to get \( |g(f(x)) - 3| < \varepsilon \). In other words, \( 0 < |x - 1| < \delta \implies |g(f(x)) - 3| < \varepsilon \). This would be incorrect. We need \( u \) to satisfy \( 0 < |u - 2| < \delta' \) in order to apply (1). So, we would need to know that \( 0 < |f(x) - 2| < \delta' \) in order to apply (1) with \( u = f(x) \). Unfortunately, we know only that \( |f(x) - 2| < \delta' \).

Having failed to write a correct proof, we might try to find examples of \( f \) and \( g \) satisfying the given conditions. To find these, we should look for examples of the sort that make our attempted proof incorrect. Note that \( f(x) = 2 \) satisfies \( |f(x) - 2| < \delta' \), but does not satisfy \( 0 < |f(x) - 2| < \delta' \). It also satisfies \( \lim_{x \to 1} f(x) = 2 \), so we might try setting \( f(x) = 2 \). The conditions on \( g \)
then become \( \lim_{u \to 2} g(u) = 3 \) and \( \lim_{x \to 1} g(2) = 42 \). In other words, \( \lim_{u \to 2} g(u) = 3 \) and \( g(2) = 42 \). These conditions are satisfied by the choice of

\[
g(x) = \begin{cases} 
3 & \text{if } x \neq 2 \\
42 & \text{if } x = 2 
\end{cases}
\]

3. Prove that \( \lim_{x \to 1} \frac{1}{x} = 1 \). Do a direct proof from the \( \varepsilon-\delta \) definition of limit.

**Solution:** When proving \( \lim_{x \to a} f(x) = L \), it is often useful to first find a relationship between \( |x - a| \) and \( |f(x) - L| \). Let us do this in the context of the given problem. If \( 0 < |x - 1| < \delta \), then

\[
\left| \frac{1}{x} - 1 \right| = \left| \frac{1-x}{x} \right| = \frac{|x-1|}{|x|} < \frac{1}{\delta}.
\]

In other words,

\[
0 < |x - 1| < \delta \implies \left| \frac{1}{x} - 1 \right| < \frac{1}{\delta}.
\]

In light of (3), we must estimate \( \frac{1}{|x|} \). To do this, suppose that \( 0 < |x - 1| < \frac{1}{2} \). One then checks that that \( x \in (\frac{1}{2}, 1) \) or \( x \in (1, \frac{3}{2}) \). In either case, \( |x| > \frac{1}{2} \), allowing us to conclude that \( \frac{1}{|x|} < 2 \). The following implication summarizes our findings.

\[
0 < |x - 1| < \frac{1}{2} \implies \frac{1}{|x|} < 2.
\]

Let us use (3) and (4) to find the desired relationship between \( |x - 1| \) and \( |\frac{1}{x} - 1| \). Indeed, if \( 0 < |x - 1| < \delta \) and \( 0 < |x - 1| < \frac{1}{2} \), then

\[
\left| \frac{1}{x} - 1 \right| = \left| \frac{1-x}{x} \right| = \frac{|x-1|}{|x|} < \frac{1}{\delta} < 2 \delta.
\]

In other words,

\[
\left( 0 < |x - 1| < \delta \text{ and } 0 < |x - 1| < \frac{1}{2} \right) \implies \left| \frac{1}{x} - 1 \right| < 2 \delta.
\]

Equivalently, we have

\[
0 < |x - 1| < \min\{\delta, \frac{1}{2}\} \implies \left| \frac{1}{x} - 1 \right| < 2 \delta.
\]
Now, let $\varepsilon > 0$ be given. Note that (5) with $\delta = \frac{\varepsilon}{2}$ reads as

$$0 < |x - 1| < \min\{\frac{\varepsilon}{2}, \frac{1}{2}\} \implies \left| \frac{1}{x} - 1 \right| < 2 \left( \frac{\varepsilon}{2} \right) = \varepsilon.$$  

(6)

By (6), $\delta = \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$ works for the given $\varepsilon$. The proof is therefore complete.

4. Prove the following theorem:

**Theorem.** Let $a \in \mathbb{R}$. Let $f$ be a function defined, at least, on an interval centered at $a$, except possibly at $a$. Let $L \in \mathbb{R}$.

IF \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \)

THEN \( \lim_{x \to a} f(x) = L \).

**Solution:** We offer two proofs of the theorem. While Proof #1 is the longer of the two, it reveals much more of the underlying intuition. In contrast, Proof #2 is more of a direct $\varepsilon - \delta$ proof.

**Proof #1:** Assume that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

We will benefit from a closer examination of the condition $0 < |x - a| < \delta$. Note that

$$0 < |x - a| < \delta \iff x \neq a \text{ and } |x - a| < \delta$$

$$\iff x \neq a \text{ and } -\delta < x - a < \delta$$

$$\iff x \neq a \text{ and } a - \delta < x < a + \delta$$

More concisely,

$$0 < |x - a| < \delta \iff (a - \delta < x < a \text{ or } a < x < a + \delta).$$  

(7)

Using (7), we replace $0 < |x - a| < \delta$ to obtain the following equivalent definition of $\lim_{x \to a} f(x) = L$:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } (a - \delta < x < a \text{ or } a < x < a + \delta) \implies |f(x) - L| < \varepsilon.$$  

(8)

The statement $(a - \delta < x < a \text{ or } a < x < a + \delta) \implies |f(x) - L| < \varepsilon$ means that if $x$ satisfies one of $a - \delta < x < a$, $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$. In
other words: if \( a - \delta < x < a \), then \(|f(x) - L| < \varepsilon\) AND if \( a < x < a + \delta \), then \(|f(x) - L| < \varepsilon\). More concisely, we have \( a - \delta < x < a \implies |f(x) - L| < \varepsilon\) AND \( a < x < a + \delta \implies |f(x) - L| < \varepsilon\). Our definition (8) of \( \lim_{x \to a} f(x) = L \) then becomes:

\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \quad a - \delta < x < a \implies |f(x) - L| < \varepsilon, \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \varepsilon.
\]

Let’s try to prove \( \lim_{x \to a} f(x) = L \) using the definition (9). Accordingly, let \( \varepsilon > 0 \) be given. According to (9), we must find \( \delta > 0 \) such that

(i) \( a - \delta < x < a \implies |f(x) - L| < \varepsilon, \) and

(ii) \( a < x < a + \delta \implies |f(x) - L| < \varepsilon. \)

Since \( \lim_{x \to a^-} f(x) = L \), we can find \( \delta_1 > 0 \) such that \( a - \delta_1 < x < a \implies |f(x) - L| < \varepsilon. \)

If we choose \( \delta \) so that \( \delta \leq \delta_1 \), then \( a - \delta_1 \leq a - \delta \). So, \( a - \delta < x < a \implies a - \delta_1 < x < a \implies |f(x) - L| < \varepsilon. \) Hence, if \( \delta \) is chosen so that \( \delta \leq \delta_1 \), then \( a - \delta < x < a \implies |f(x) - L| < \varepsilon. \) In other words, if \( \delta \leq \delta_1 \), then (i) is satisfied.

Since \( \lim_{x \to a^+} f(x) = L \), we can find \( \delta_2 > 0 \) such that \( a < x < a + \delta_2 \implies |f(x) - L| < \varepsilon. \)

If we choose \( \delta \) so that \( \delta \leq \delta_2 \), then \( a + \delta \leq a + \delta_2 \). So, \( a < x < a + \delta \implies a < x < a + \delta_2 \implies |f(x) - L| < \varepsilon. \) Hence, if \( \delta \) is chosen so that \( \delta \leq \delta_2 \), then \( a < x < a + \delta_2 \implies |f(x) - L| < \varepsilon. \) In other words, if \( \delta \leq \delta_2 \), then (ii) is satisfied.

In light of our findings above, if \( \delta \leq \delta_1 \) and \( \delta \leq \delta_2 \), then (i) and (ii) are satisfied. In particular, \( \delta := \min\{\delta_1, \delta_2\} \) satisfies (i) and (ii). We have therefore shown \( \lim_{x \to a} f(x) = L \) using the equivalent definition (9).

**Proof #2:** Assume that \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L. \) Using the \( \varepsilon - \delta \) definition of the limit, we must verify that \( \lim_{x \to a} f(x) = L. \)

Let \( \varepsilon > 0 \) be given. Our objective is to find some \( \delta > 0 \) such that

\[
0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.
\]

Since \( \lim_{x \to a^-} f(x) = L \), there exists \( \delta_1 > 0 \) such that

\[
a - \delta_1 < x < a \implies |f(x) - L| < \varepsilon. \tag{10}
\]
Similarly, the condition \( \lim_{x \to a^+} f(x) = L \) implies that there exists \( \delta_2 > 0 \) such that

\[
a < x < a + \delta_2 \implies |f(x) - L| < \varepsilon.
\]  

(11)

Now, set \( \delta := \min\{\delta_1, \delta_2\} \). We claim that \( 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \). To this end, assume that \( 0 < |x - a| < \delta \). Using (7), we see that \( a - \delta < x < a \) or \( a < x < a + \delta \). Let us consider these two cases separately.

Case #1: \( a - \delta < x < a \): Since \( \delta \leq \delta_1 \), it follows that \( a - \delta_1 \leq a - \delta \). Hence, it is true that \( a - \delta_1 < x < a \). Using (10), we conclude that \( |f(x) - L| < \varepsilon \).

Case #2: \( a < x < a + \delta \): Since \( \delta \leq \delta_2 \), it follows that \( a + \delta \leq a + \delta_2 \). Hence, it is true that \( a < x < a + \delta_2 \). Using (11), we conclude that \( |f(x) - L| < \varepsilon \).

In light of the above, it follows that \( |f(x) - L| < \varepsilon \). Hence, we have shown that

\[
0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.
\]

This completes the proof.