Selective Ultrafilters on FIN

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Abstract

We consider selective ultrafilters on the collection FIN of all finite nonempty subsets of \( \mathbb{N} \). If countable-support side-by-side Sacks forcing is applied, then every selective ultrafilter in the ground model generates a selective ultrafilter in the extension. We also show that selective ultrafilters localize the Parametrized Milliken Theorem, and that selective ultrafilters are Ramsey.

0 Introduction

In [2] and [11], Baumgartner and Laver showed that selective ultrafilters on \( \mathbb{N} \) are preserved under both side-by-side and iterated Sacks forcing. That is, if Sacks forcing is applied then every selective ultrafilter on \( \mathbb{N} \) in the ground model has its upward closure being a selective ultrafilter on \( \mathbb{N} \) in the extension. Consider the collection FIN of all finite nonempty subsets of \( \mathbb{N} \). Milliken [14] introduced the collection \( \text{FIN}^{[\infty]} \) of all infinite block-sequences of members of FIN, a more powerful topological Ramsey space ([20]) than the Ellentuck space \( \mathbb{N}^{[\infty]} \), and strengthened various generalizations of Ramsey’s theorem.

The original conjecture that many topological Ramsey spaces have an ultrafilter associated to them analogous to the way selective ultrafilters on \( \mathbb{N} \) are related to the Ellentuck space was made by Todorcevic in the 1990’s when he proved using large cardinals that selective ultrafilters are generic over \( L(\mathbb{R}) \) for the poset \( \mathcal{P}(\mathbb{N})/\text{FIN} \) ([6]). In search of a notion of selectivity on FIN that captures the preservation under countable-support side-by-side Sacks forcing, we define selective ultrafilters on FIN in such a way that the combinatorial properties of \( U \)-trees can be easily applied. The notion of \( U \)-trees was introduced by Blass in [3]. Our definition turns out to be equivalent to Blass and Hindman’s stable ordered-union ultrafilters in [4], and it is a special case of Mijares’ general notion of selective ultrafilters for topological Ramsey spaces in [13]. It also coincides with a special case of Di Prisco, Mijares and Nieto’s definition in [5], which they used to prove complete combinatorics. It is also worth noting that Fernández-Breton and Hrušák [9] showed the existence of such selective ultrafilters on FIN in Sacks’ model assuming a weak diamond principle.

To show that selective ultrafilters on FIN are preserved under Sacks forcing, we first prove that a selective ultrafilter localizes the following theorem in [20, Thm. 5.45]:

**Theorem 0.1** (Parametrized Milliken Theorem [20]). For every finite Souslin-measurable colouring of \( \text{FIN}^{[\infty]} \times (2^\omega)^\omega \) there exist \( B \in \text{FIN}^{[\infty]} \) and a sequence \((P_i)_{i<\omega}\) of nonempty perfect subsets of \( 2^\omega \) such that \( [B]^{[\infty]} \times \prod_{i<\omega} P_i \) is monochromatic.

This is done by defining a property of subsets of \( \text{FIN}^{[\infty]} \times (2^\omega)^\omega \), called perfectly \( U \)-Ramsey, for selective ultrafilters \( U \), and showing that this property is preserved under the Souslin operation. The result of Mathias [12] and Silver [19], that every analytic set is Ramsey, was strengthened by Ellentuck [8] to all analytic sets with respect to the finer Ellentuck topology. However, Ramsey properties are not often captured by a natural topology. We follow a similar procedure but rely more on the combinatorial properties of \( U \)-trees.

In Section 1, we recall some definitions related to the Milliken space \( \text{FIN}^{[\infty]} \), perfect subsets of \( 2^\omega \) and Sacks forcing. In Section 2, we prove the following localized version of the Parametrized Milliken Theorem:

**Theorem 0.2** (Local Parametrized Milliken Theorem). Let \( U \) be a selective ultrafilter on FIN. For every finite Souslin-measurable colouring of \( \text{FIN}^{[\infty]} \times (2^\omega)^\omega \), there exist \( [X] \in U \) and a sequence \((P_i)_{i<\omega}\) of nonempty perfect subsets of \( 2^\omega \) such that \( [\emptyset, X] \times \prod_{i<\omega} P_i \) is monochromatic.

2010 Mathematics Subject Classification. Primary 05D10; Secondary 03E02, 03E40.
In Section 3, we show that selective ultrafilters are preserved after adding arbitrarily many Sacks reals by countable-support side-by-side Sacks forcing.

**Theorem 0.3.** Let $\kappa$ be an infinite cardinal, and $P_\kappa$ be countable-support side-by-side Sacks forcing adding $\kappa$ Sacks reals. Let $U$ be a selective ultrafilter on $\text{FIN}$ in the ground model, and $\check{V}$ a name for the upward closure $\{\check{Y} \subseteq \text{FIN} : \exists [\check{X}] \in U \mid [\check{X}] \subseteq \check{Y}\}$ of $U$. Then $\Vdash_{P_\kappa} \check{V}$ is a selective ultrafilter on $\text{FIN}$.

In Section 4, we prove that every selective ultrafilter on $\text{FIN}$ is Ramsey. This is in contrast to the case in the spaces $R_\alpha$, as shown by Trujillo in [21]. The spaces $R_\alpha$, for countable ordinals $\alpha$, are topological Ramsey spaces constructed by Dobrinen and Todorcevic in [7].

I am grateful to Professor Stevo Todorcevic for his guidance. I would also like to thank Mr. Xiao Qing Zheng for his uncountable support.

# 1 Preliminaries

A (non-principal) ultrafilter on some set $S$ is a collection $U$ of subsets of $S$ with the following properties. For subsets $M,N,A,B$ of $S$,

(a) $S \in U$ but $\{x\} \notin U$ for all $x \in S$,

(b) $M \subseteq N$ and $M \in U$ implies that $N \in U$,

(c) $M = A \cup B$ and $M \in U$ implies that $A \in U$ or $B \in U$,

(d) $M \in U$ and $N \in U$ implies that $M \cap N \in U$.

## 1.1 The Milliken space $\text{FIN}^{[\omega]}$

Let $\text{FIN}$ be the collection of all finite nonempty subsets of $\mathbb{N}$. We will concentrate on the case $S = \text{FIN}$. For elements $x,y$ in $\text{FIN}$, by $x < y$ we mean that $\max(x) < \min(y)$. Listed below are some definitions and notation for us to refer back to.

**Definition 1.1.** A block-sequence is a sequence of elements in $\text{FIN}$ of the form $X = (x_n) = \{x_1, x_2, \ldots\} \subseteq \text{FIN}$, where each $x_n$ is called a block, and $x_n < x_{n+1}$ for all $n$. Let $\text{FIN}^{[\omega]}$ denote the collection of all infinite block-sequences, $\text{FIN}^{[<\omega]}$ the collection of all finite block-sequences. For $a \in \text{FIN}^{[<\omega]}$, let the length $|a|$ of $a$ be the number of blocks in $a$. Let $\text{FIN}^{[n]}$ the collection of block-sequences of length $n$.

**Notation.** For $X = (x_n) \in \text{FIN}^{[\omega]}$, $k < \omega$ and $a \in \text{FIN}^{[<\omega]}$, let $X/a = \{x \in X : \cup a < x\}$. For $X = (x_n) \in \text{FIN}^{[\omega]}$, let $[X]$ denote the sublattice of $\text{FIN}$ generated by $X$, that is,

$$[X] = \{x_{n_0} \cup \cdots \cup x_{n_k} : k < \omega, n_0 < \cdots < n_k < \omega\}.$$

When we say $[X]$ is an element of an ultrafilter, we tacitly understand that $X \in \text{FIN}^{[\omega]}$.

**Definition 1.2.** For two (finite or infinite) block-sequences $X = (x_n)$ and $Y$, we say $X$ is a block-subsequence of $Y$, and write $X \leq Y$, if $x_n \in [Y]$ for all $n$. Let $[0,Y]$ denote the collection of all infinite block-subsequences of $Y$. For an infinite block-sequence $X$ and a finite block-subsequence $a \leq X$, let $[a,X] = \{Y \in \text{FIN}^{[\omega]} : a \subseteq Y \leq X\}$.

Unless stated otherwise, we always assume the metric topology for $\text{FIN}^{[\omega]}$ with the first-difference metric $\rho$ as follows. For $X = (x_n)_{n=1}^{\infty}$ and $Y = (y_n)_{n=1}^{\infty}$,

$$\rho(X,Y) = \frac{1}{k} \text{ where } k = \min\{n : x_n \neq y_n\}.$$

This induces a topology on $\text{FIN}^{[\omega]}$ with open balls of the following form.

$$\mathcal{B}(A,\frac{1}{k}) = \{X \in \text{FIN}^{[\omega]} : \rho(X,A) < \frac{1}{k}\}$$

$$= \{X = (x_n) \in \text{FIN}^{[\omega]} : x_n = a_n \text{ for all } n \leq k\}$$
for $A = (a_n)_{n=1}^\infty \in \text{FIN}^{[\infty]}$. We may also write the open balls as

$$[a] := \{X = (x_n) \in \text{FIN}^{[\infty]} : x_n = a_n \text{ for all } n \leq k\} \text{ for } a = (a_n)_{n=1}^k \in \text{FIN}^{<\infty}.$$ 

1.2 The spaces $2^\omega$ and $(2^\omega)^\omega$, and Sacks forcing

Let $2^\omega$ have the product topology. We identify $(2^\omega)^\omega$ and $2^{\omega \times \omega}$, each equipped with the product topology. Let $S^q$ be the set of all finite 01-sequences, that is, $S^q = \bigcup \{t : n \in \omega\}$. For $s \in S^q$, we denote the length of $s$ by $|s|$. For $n < \omega$ and $f \in 2^{\omega(n)}$, by $f|n$ we mean the sequence obtained by restricting $f$ to its first $n$ elements. If $n > |f|$, let $f|n = f$. For $s \in S^q$ and $f \in 2^{\omega(n)}$, $s \subseteq f$ means that $s$ is an initial segment of $f$. For $s, t \in S^q$, we say $s, t \in S^q$ are comparable if $s \subseteq t$ or $t \subseteq s$; otherwise we say $s$ and $t$ are incomparable.

**Definition 1.3.** [1]. A nonempty set $p \subseteq S^q$ is a tree if for every $s \in p$ and $n < \omega$, $s|n \in p$. Moreover, a tree $p$ is perfect if for every $s \in p$ there exist incomparable $t, u \in p$ such that $s \subseteq t$ and $s \subseteq u$. In particular, every perfect tree is infinite.

**Notation.** For a tree $p \subseteq S^q$, let $[p]$ denote the set of all infinite branches of $p$:

$$[p] = \{f \in 2^\omega : f|n \in p \text{ for all } n \in \omega\}.$$ 

For a closed set $F \subseteq 2^\omega$, let $T_F$ be the tree whose infinite branches are the elements of $F$:

$$T_F = \{s \in S^q : s \subseteq f \text{ for some } f \in F\}.$$ 

**Lemma 1.4.** [1]. A closed set $F \subseteq 2^\omega$ is perfect if and only if the tree $T_F$ is a perfect tree.

**Definition 1.5.** [18]. Sacks forcing $\mathcal{P}$ is the set of all perfect trees, with the ordering $p \leq q$ if $p \subseteq q$. In particular, $p \leq q$ if and only if $[p] \subseteq [q]$.

**Definition 1.6.** Let $p \in \mathcal{P}$ and $s \in p$. The branching level of $s$ in $p$ is the number of branchings below $s$ in the tree $p$, namely

$$|\{i < |s| : \exists t \in p ((|t| > i) \land (t|i = s|i) \land (t| (i+1) \neq s|(i+1)))\}|.$$ 

The $n^{th}$ branching level $l(n, p)$ of $p$ is the set of all $s \in p$ which have branching level $n$ and are minimal with this property.

**Definition 1.7.** [1]. For $p, q \in \mathcal{P}$, we write $p \leq^n q$ if $p \leq q$ and $l(n, p) = l(n, q)$.

**Definition 1.8.** [1]. For $p \in \mathcal{P}$ and $s \in p$, let $p|s = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}$. Notice that $p|s \in \mathcal{P}$.

Sacks forcing $\mathcal{P}$ adds one Sacks real ([18]). In this paper, we consider countable-support side-by-side Sacks forcings $\mathcal{P}_\kappa$ which adds $\kappa$ Sacks reals, where $\kappa$ is an arbitrary infinite cardinal.

**Definition 1.9.** [1]. Let $\kappa$ be an infinite cardinal. Let $\mathcal{P}_\kappa$ be the set of all functions $p : \kappa \to \mathcal{P}$ such that for all but countably many $\alpha < \kappa$, $p(\alpha) = 2^{\omega(\alpha)}$. The ordering on $\mathcal{P}_\kappa$ is given by $p \leq q$ if $p(\alpha) \subseteq q(\alpha)$ for every $\alpha < \kappa$. The support of $p \in \mathcal{P}_\kappa$ is $\text{supp}(p) = \{\alpha < \kappa : p(\alpha) \neq 2^{\omega(\alpha)}\}$.

The following definitions lead us to the Fusion Lemma 1.12. The method of fusion is used throughout this paper.

**Definition 1.10.** [1]. For $p \in \mathcal{P}_\kappa$, $n \in \omega$ and a finite set $F \subseteq \kappa$, let

$$l(F, n, p) = \{\sigma : (\text{dom}(\sigma) = F) \land ((\forall \alpha \in F)(\sigma(\alpha) \in l(n, p(\alpha))))\}.$$ 

We write $p \leq^{F,n} q$ if $p \leq q$ and $l(F, n, p) = l(F, n, q)$.

**Definition 1.11.** [1]. Let $p \in \mathcal{P}_\kappa$, and $F$ be a finite subset of $\kappa$. Let $\sigma$ be a function with $\text{dom}(\sigma) = F$ and $\sigma(\alpha) \in p(\alpha)$ for every $\alpha \in F$. We define $p|\sigma \in \mathcal{P}_\kappa$ as follows. For $\alpha < \kappa$,

$$p|\sigma(\alpha) = \begin{cases} p(\alpha) & \text{for } \alpha \notin F; \\ p(\alpha)|\sigma(\alpha) & \text{for } \alpha \in F. \end{cases}$$
Lemma 1.12 (Fusion Lemma). [1, Lem. 1.6] Suppose \( \langle p_n : n \in \omega \rangle \subseteq \mathcal{P}_\kappa \) and \( \langle (F_n, m_n) : n \in \omega \rangle \) are sequences such that \( F_n \subseteq F_{n+1} \subseteq \kappa \) are finite, \( m_n \leq m_{n+1} < \omega \), \( \lim m_n = \infty \) and \( p_{n+1} \leq F_n, m_n \) \( p_n \) for each \( n \). Suppose also that \( \bigcup \{ F_n : n \in \omega \} \supseteq \bigcup \{ \supp(p_n) : n \in \omega \} \). Define \( q : \kappa \to \mathcal{P} \) as follows:

\[
q(\alpha) = \bigcap_{\beta < \alpha} p_\beta(\alpha) \quad \text{for } \alpha < \kappa.
\]

Then \( q \in \mathcal{P}_\kappa \) and \( q \leq F_n, m_n \) \( p_n \) for each \( n \).

If \( \langle p_n : n \in \omega \rangle \subseteq \mathcal{P}_\kappa \), \( \langle (F_n, m_n) : n \in \omega \rangle \) and \( q \) are as above, then we say that \( \langle p_n : n \in \omega \rangle \) is a fusion sequence and \( q \) is the fusion of the sequence.

Definition 1.13. For \( p \in \mathcal{P}_\omega \), we define the set \( [p] \subseteq (2^\omega)^\omega \) to be \( [p] = \prod_{i<\omega}[p(i)]. \)

Recall that \( 2^\omega \) and \( (2^\omega)^\omega \) are assumed to have the product topology. The space \( 2^\omega \) has basic open sets of the form \( [s] = \{ f \in 2^\omega : s \subseteq f \} \). Thus, for \( p \in \mathcal{P} \) and \( f \in [p] \), every initial segment \( s \subseteq f \) is associated with an open set \( [s] \) in \( 2^\omega \). We develop a similar notion for \( p \in \mathcal{P}_\omega \) and \( (2^\omega)^\omega \). Let \( p \in \mathcal{P}_\omega \). An element \( \epsilon \in [p] \) is a function \( \epsilon : \omega \times \omega \to \omega \) with \( \epsilon(j, i) \in [p(i)] \) for every \( i, j < \omega \). For a fixed \( i \in \omega \), we may think of \( \{ (\epsilon(j, i))_{j<\omega} \} \) as a function in \( i \), and hence an infinite branch in the \( i \)th perfect tree \( p(i) \) of \( p \). We define a finite partial function \( \sigma : \omega \to 2^{<\omega} \) to be a pre-initial segment of \( \epsilon \) if for every \( i \in \dom(\sigma) \), \( \sigma(i) \) is an initial segment of \( (\epsilon(j, i))_{j<\omega} \):

Definition 1.14. Let \( \epsilon \in (2^\omega)^\omega \). A partial function \( \sigma : \omega \to 2^{<\omega} \) with finite domain is a pre-initial segment of \( \epsilon \) if \( \sigma(i) \in (\epsilon(j, i))_{j<\omega} \) for every \( i \in \dom(\sigma) \).

Then \( (2^\omega)^\omega \) has basic open subsets of the form \( [\sigma] = \{ \epsilon \in (2^\omega)^\omega : \sigma \text{ is a pre-initial segment of } \epsilon \} \) for finite partial functions \( \sigma : \omega \to 2^{<\omega} \), or equivalently, for \( \sigma \in 2^{(\omega \times \omega) \times (\omega \times \omega)} \). Also throughout this paper, \( \FIN^\infty \times 2^\omega \) and \( \FIN^\infty \times (2^\omega)^\omega \) have the product topology, where \( \FIN^\infty \) has the first-difference metric topology unless otherwise stated. Therefore basic open sets in \( \FIN^\infty \times (2^\omega)^\omega \) are of the form \( [a] \times [\sigma] \) where \( a \in \FIN^{<\infty} \) and \( \sigma \in 2^{(\omega \times \omega) \times (\omega \times \omega)} \).

2 Selective ultrafilters are localizing

Definition 2.1. [4, 13] An ultrafilter \( \mathcal{U} \) on \( \FIN \) is selective if

1. \( \mathcal{U} \) is generated by elements of the form \([X] \) where \( X \in \FIN^{<\infty} \);

2. for every set \( \{ [X_a] : a \in \FIN^{<\infty} \} \) of elements in \( \mathcal{U} \), there exists \([X] \in \mathcal{U} \) such that for all \( a \leq X \), \( X/a \subseteq [X_a] \).

Note that by property (1) of selectivity, if \([X] \in \mathcal{U} \), then for every \( a \in \FIN^{<\infty} \), \([X/a] \in \mathcal{U} \). Moreover, a selective ultrafilter \( \mathcal{U} \) is idempotent with operation \( \cup \).

Lemma 2.2. Given a selective ultrafilter \( \mathcal{U} \) on \( \FIN \), let \( \mathcal{U} \cup \mathcal{U} = \{ A \subseteq \FIN : \{ x \in \FIN : \{ y \in \FIN : x \cup y \in A \} \in \mathcal{U} \} \subseteq \mathcal{U} \} \). Then \( \mathcal{U} = \mathcal{U} \cup \mathcal{U} \).

Proof. We prove that \( \mathcal{U} \subseteq \mathcal{U} \cup \mathcal{U} \) and equality follows since \( \mathcal{U} \cup \mathcal{U} \) is also an ultrafilter. Consider \( [B] \in \mathcal{U} \), and let \( B = (b_n)_{n<\omega} \). We show that \( [B] \in \mathcal{U} \cup \mathcal{U} \). Let \( X = \{ x : \{ y : x \cup y \in [B] \} \in \mathcal{U} \} \). It is sufficient to prove that \( X \in \mathcal{U} \). We show this by proving that \( [B] \subseteq X \) and using the property that \( \mathcal{U} \) is closed under superset.

For an arbitrary \( x \in [B] \), let \( x = b_{n_0} \cdot \cdots \cdot b_{n_l} \) with \( n_0 < \cdots < n_l \). Then \( [B/x] \subseteq \{ y : x \cup y \in [B] \} \). \( [B] \in \mathcal{U} \), so \( [B/x] \in \mathcal{U} \), and hence \( \{ y : x \cup y \in [B] \} \in \mathcal{U} \), that is \( x \in X \). As \( x \in [B] \) was chosen arbitrary, we have \( [B] \subseteq X \) as required.

Definition 2.3. An ultrafilter \( \mathcal{U} \) on \( \FIN \) is localizing if for every Souslin-measurable set \( B \) of \( \FIN^{\infty} \times (2^\omega)^\omega \) there exist \([B] \in \mathcal{U} \) and \( p \in \mathcal{P}_\omega \) such that \([0, B] \times [p] \subseteq B \) or \([0, B] \times [p] \cap B = \emptyset \).

A localizing ultrafilter localizes the Parametrized Milliken Theorem 0.1. In this section we aim to prove that every selective ultrafilter is localizing. We first consider open subsets of \( \FIN^{\infty} \times (2^\omega)^\omega \) in subsection 2.1, then generalize the result to Souslin-measurable subsets in subsection 2.2. From now on in this section, we fix an arbitrary selective ultrafilter \( \mathcal{U} \) on \( \FIN \).
2.1 Open subsets of $\text{FIN}^{[\omega]} \times (2^\omega)^\omega$

Ultra-Ramsey theory plays a role in the proof. So we adapt definitions in [20, Section 7.2] for the space $\text{FIN}^{[\omega]}$. Consider $\text{FIN}^{[\omega]}$ as a tree ordered by end-extension $\subseteq$ with root $\emptyset$. From now on in this section, by a tree we mean a subtree of $\text{FIN}^{[\omega]}$ that is downward closed, namely, if the subtree contains an element $a$, then it contains every initial segment $b \subseteq a$.

**Notation.** For a tree $T$, we denote the collection of all infinite branches of $T$ by $[T]$, that is,

$$[T] = \left\{ A = (a_i)_{i=1}^{\infty} \subseteq \text{FIN}^{[\omega]} : \{a_1, \ldots , a_n\} \in T \text{ for all } n \in \omega \right\}.$$

Let the maximal node of $T$ that is comparable with every other node of $T$ be denoted by $st(T)$, called the stem of $T$.

**Definition 2.4.** [20, Def. 7.28]. A $\mathcal{U}$-tree $T$ is a tree such that, for all $t \in T$ with $st(T) \leq t$, $\{x \in \text{FIN} : t \subseteq x \cup \{x\} \in T\} \in \mathcal{U}$. For two $\mathcal{U}$-trees $T$ and $T'$, we say $T'$ is a pure refinement of $T$, and write $T' \leq 0 T$, if $T' \subseteq T$ and $st(T') = st(T)$.

**Definition 2.5.** [20, Def. 7.35]. A subset $G$ of $\text{FIN}^{[\omega]}$ is $\mathcal{U}$-open if for every $A \in G$ there is a $\mathcal{U}$-tree $T$ such that $A \subseteq [T]$ and $[T] \subseteq G$.

The collection of all $\mathcal{U}$-open subsets of $\text{FIN}^{[\omega]}$ is a topology on $\text{FIN}^{[\omega]}$ with basis $[T]$ for $\mathcal{U}$-trees. This $\mathcal{U}$-topology includes the metric topology on $\text{FIN}^{[\omega]}$.

**Definition 2.6.** [20, Def. 7.37]. A subset $\mathcal{X}$ of $\text{FIN}^{[\omega]}$ is $\mathcal{U}$-Ramsey if for every $\mathcal{U}$-tree $T$ there is a pure refinement $T' \leq 0 T$ such that $[T'] \subseteq \mathcal{X}$ or $[T'] \cap \mathcal{X} = \emptyset$.

The proof of Theorem 7.42 in [20] can be readily adapted to give us the following.

**Lemma 2.7.** [20, Thm. 7.42]. If $\mathcal{X} \subseteq \text{FIN}^{[\omega]}$ is (metrically) Souslin-measurable, then $\mathcal{X}$ is $\mathcal{U}$-Ramsey.

**Lemma 2.8.** For a $\mathcal{U}$-tree $T$ with stem $a$, there exists $[X] \in \mathcal{U}$ with $a \subseteq X$ such that $[a,X] \subseteq [T]$. Conversely, for $a \leq X$ with $[X] \in \mathcal{U}$, there exists a $\mathcal{U}$-tree $T$ with stem $a$ such that $[T] \subseteq [a,X]$.

**Proof.** First we assume $T$ is a $\mathcal{U}$-tree with stem $a$. Since $T$ is a $\mathcal{U}$-tree, for each $b > a$ or $b = \emptyset$ with $a \cup b \in T$ the set $A_b = \{ x \in \text{FIN} : a \cup b \subseteq a \cup b \cup \{x\} \in T\}$ is in $\mathcal{U}$. So by property (1) of selectivity, there exists $[X_b] \in \mathcal{U}$ with $[X_b] \subseteq A_b$. On the other hand, for each $b \in \text{FIN}^{[\omega]} \setminus \{ b : a \subseteq a \cup b \in T\}$, let $[X_b] = \text{FIN}^{[\omega]}$. Now we have $\{ [X_b] : b \in \text{FIN}^{[\omega]} \} \subseteq \mathcal{U}$. By property (2) of selectivity, there exists an $[X_\infty] \in \mathcal{U}$ such that $X_\infty \cup b \subseteq X_\infty$ for every $b \leq X_\infty$.

Let $X = a \cup (X_\infty/a)$. Then $[X] \in \mathcal{U}$. We check that $[a,X] \subseteq [T]$. Let $Y = a \cup (y_n)_{n<\omega} \subseteq [a,X]$. We show $a \cup \{y_1, \ldots , y_n\} \in T$ for every $n \in \omega$ by induction. For $n = 0$, it is $a$, the stem of $T$. Suppose $n = k + 1$. By the induction hypothesis, $a \cup \{y_1, \ldots , y_k\} \in T$. Then $y_{k+1} \in \{X/(a \cup \{y_1, \ldots , y_k\}) = X_\infty \setminus \{y_1, \ldots , y_k\}\} \subseteq X_\infty \cup \{y_1, \ldots , y_k\} \subseteq A_\infty$. So by the definition of $A_\infty$, $a \cup \{y_1, \ldots , y_k, y_{k+1}\} \in T$. Thus, if $Y \in [a,X]$ then $Y \subseteq [T]$. Hence $[a,X] \subseteq [T]$ as required.

Conversely, let $[X] \in \mathcal{U}$ and $a \leq X$. The set $T := \{ b \in \text{FIN}^{[\omega]} : b \subseteq a \} \cup \{ b \in \text{FIN}^{[\omega]} : a \subseteq b \leq X\}$ is a $\mathcal{U}$-tree with stem $a$. This is because, for every $b \in T$ with $a \subseteq b$, $\{ x \in \text{FIN} : b \subseteq b \cup \{x\} \in T\} = [X/b] \in \mathcal{U}$. □

**Definition 2.9.** For a $\mathcal{U}$-tree $T$ and $n \in \omega$, let $l_n(T)$ be the $n$th level of the tree $T$ above its stem:

$$l_n(T) = \{ b \in T : |b| = |st(T)| + n \}.$$

For $[X] \in \mathcal{U}$ and $a \leq X$, let $l_n[a,X] = \{ b \in \text{FIN}^{[a \cup [n]+n]} : a \subseteq b \leq X\}$.

**Theorem 2.10.** For every Souslin-measurable subset $\mathcal{X}$ of $\text{FIN}^{[\omega]}$, there exists $[X] \in \mathcal{U}$ such that $[\emptyset,X] \subseteq \mathcal{X}$ or $[\emptyset,X] \cap \mathcal{X} = \emptyset$.

**Proof.** By Lemma 2.7, $\mathcal{X}$ is $\mathcal{U}$-Ramsey. So for the $\mathcal{U}$-tree $\text{FIN}^{[\omega]}$, there exists a pure refinement $T \leq 0 \text{FIN}^{[\omega]}$ such that $[T] \subseteq \mathcal{X}$ or $[T] \cap \mathcal{X} = \emptyset$. By Lemma 2.8, there exists $[X] \in \mathcal{U}$ with $[\emptyset,X] \subseteq [T]$. □
Now we aim to prove that for an open set $\mathcal{O}$ of $\text{FIN}^{[\omega]} \times (2^\omega)^\omega$ in the product topology, the set $\{X \in \text{FIN}^{[\omega]} : \exists p \in \mathcal{P}_\omega (([0, X] \times [p] \subseteq \mathcal{O})$ or $([0, X] \times [p] \cap \mathcal{O} = \emptyset)\}$ is Souslin-measurable, after which we could apply Theorem 2.10. For this, we define uniform families in $\text{FIN}^{[\omega]}$. It is analogous to Pudlák and Rödl’s definition of uniform family in the Ellentuck space in [17].

**Notation.** For $\mathcal{S} \subseteq \text{FIN}^{[\omega]}$ and $a \in \text{FIN}^{[\omega]}$, let $\mathcal{S}_{[a]} = \{y : a \subseteq y \in \mathcal{S}\}$.

**Definition 2.11.** [17]. Let $\gamma$ be a countable ordinal and $\mathcal{S} \subseteq \text{FIN}^{[\omega]}$. Let $X \in \text{FIN}^{[\omega]}$ and $b \leq X$. The notion of $\gamma$-uniform is defined recursively as follows. The family $\mathcal{S}$ is $\gamma$-uniform on $[b, X]$ if

- $\gamma = 0$ and $\mathcal{S} = \{b\}$;
- $\gamma = \beta + 1$, $b \notin \mathcal{S}$ and for each $a \in l_1[b, X]$, $\mathcal{S}_{[a]}$ is $\beta$-uniform on $[a, X]$; or
- $\gamma$ is a limit ordinal, $b \notin \mathcal{S}$ and there exists a sequence $(\gamma_a)_{a \in l_1[b, X]}$ of ordinals with $\bigcup_{a \in l_1[b, X]} \gamma_a = \gamma$

such that for each $a \in l_1[b, X]$, $\mathcal{S}_{[a]}$ is $\gamma_a$-uniform on $[a, X]$.

We say $\mathcal{S}$ is uniform on $[b, X]$ if it is $\gamma$-uniform on $[b, X]$ for some $\gamma$.

For example, for $n < \omega$, the only $n$-uniform family on $[b, X]$ is $l_n[b, X]$. On the other hand, for each $k < \omega$, the family $\mathcal{S} = \{y = (y_n) \in \text{FIN}^{[\omega]} : \exists b \subseteq y \subseteq X \land (|y| = \text{max}(y_{|b|+1}) + k)\}$ is $\omega$-uniform on $[b, X]$, and $T = \{y = (y_n) \in \text{FIN}^{[\omega]} : \exists b \subseteq y \subseteq X \land (|y| = \text{max}(y_{|b|+2}) + k)\}$ is $(\omega + 1)$-uniform on $[b, X]$.

Recall from [20, Def. 1.12] that a family $\mathcal{F}$ of finite block-sequences is Nash-Williams if $x \subseteq y$ for every distinct pair $x, y \in \mathcal{F}$. The family $\mathcal{F}$ is a front on $[b, X]$ if it is Nash-Williams and every element of $[b, X]$ has an initial segment in $\mathcal{F}$. The definition of rank of a barrier in [20, Def. 1.24] can be used for fronts as well.

**Definition 2.12.** [20, Def. 1.24]. Let $\mathcal{F}$ be a front on $[b, X]$. The set $T(\mathcal{F}) = \{x : \exists y \in \mathcal{F}, x \subseteq y\}$ is the $\subseteq$-downward closure of $\mathcal{F}$. Let $\rho_\mathcal{F}(x) = \sup\{\rho_\mathcal{F}(s) + 1 : (t \supseteq s) \land (s \in T(\mathcal{F}))\}$, where sup $\emptyset$ is defined to be 0. Then the rank of $\mathcal{F}$ on $[b, X]$ is $\rho(\mathcal{F}) = \rho_\mathcal{F}(b)$.

**Lemma 2.13.** If $\mathcal{S}$ is uniform on $[b, X]$, then it is a front on $[b, X]$.

**Proof.** Let $\mathcal{S}$ be $\gamma$-uniform on $[b, X]$. We prove by induction on $\gamma$. If $\gamma = 0$, then $\mathcal{S} = \{b\}$ and the conclusion holds. So we assume $\gamma > 0$. By the definition of $\gamma$-uniform, for each $a \in l_1[b, X]$, $\mathcal{S}_{[a]}$ is $\beta$-uniform for some $\beta < \gamma$. So by the induction hypothesis, $\mathcal{S}_{[a]}$ is a front on $[a, X]$. Then $\mathcal{S} = \bigcup_{a \in l_1[b, X]} \mathcal{S}_{[a]}$ is a front on $[b, X]$.

**Lemma 2.14.** If $\mathcal{F}$ is a front on $[b, X]$, then there exists a uniform system $\mathcal{S}$ on $[b, X]$ such that every $s \in \mathcal{S}$ has an initial segment in $\mathcal{F}$.

**Proof.** We induct on the rank of $\mathcal{F}$. If $\rho(\mathcal{F}) = 0$ then $\mathcal{F} = \{b\}$ itself is a uniform system on $[b, X]$. So we assume $\rho(\mathcal{F}) > 0$. For each $a \in l_1[b, X]$, the family $\mathcal{F}_{[a]} = \{s : a \subseteq s \subseteq \mathcal{F}\}$ is a front on $[a, X]$ of a smaller rank. By the induction hypothesis, there exists a countable ordinal $\gamma_a$ and a $\gamma_a$-uniform family $\mathcal{S}_a$ on $[a, X]$ such that every element in $\mathcal{S}_a$ has an initial segment in $\mathcal{F}_{[a]}$. Then the family $\mathcal{S} = \bigcup_{a \in l_1[b, X]} \mathcal{S}_a$ is a $(\bigcup_{a \in l_1[b, X]} \gamma_a)$-uniform on $[b, X]$ and every element of $\mathcal{S}$ has an initial segment in $\mathcal{F}$.

**Theorem 2.15.** For every open set $\mathcal{O} \subseteq \text{FIN}^{[\omega]} \times (2^\omega)^\omega$ there exist $[X] \in \mathcal{U}$ and $p \in \mathcal{P}_\omega$ such that $[0, X] \times [p] \subseteq \mathcal{O}$ or $([0, X] \times [p]) \cap \mathcal{O} = \emptyset$.

**Proof.** Let $\mathcal{X}_0 = \{X \in \text{FIN}^{[\omega]} : (\exists p \in \mathcal{P}_\omega) \{([0, X] \times [p]) \cap \mathcal{O} = \emptyset\}\}$, and $\mathcal{X}_1 = \{X \in \text{FIN}^{[\omega]} : (\exists p \in \mathcal{P}_\omega) \{([0, X] \times [p]) \subseteq \mathcal{O}\}\}$.

Let $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$. We claim that $\mathcal{X}_0$ and $\mathcal{X}_1$ are Souslin-measurable, hence so is $\mathcal{X}$. Then applying Theorem 2.10 to the set $\mathcal{X} \subseteq \text{FIN}^{[\omega]}$, we get $[X] \in \mathcal{U}$ such that $[0, X] \subseteq \mathcal{X}$ or $[0, X] \cap \mathcal{X} = \emptyset$. If $[0, X] \subseteq \mathcal{X}$ then there exists $p \in \mathcal{P}_\omega$ such that $X$ and $p$ satisfies the theorem. Otherwise, $[0, X] \cap \mathcal{X} = \emptyset$, we would have the
following contradiction. The set $O \cap ([0, X] \times (2^\omega)^{<\omega})$ is open in $[0, X] \times (2^\omega)^{<\omega}$ in the subspace topology. By the Parametrized Milliken Theorem 0.1, there exist $Y \in [0, X]$ and $q \in \mathcal{P}_\omega$ such that $[0, Y] \times [p] \subseteq O$ or $((0, Y] \times [p]) \cap O = \emptyset$. Then $Y \in \mathcal{X}$, contradicting $Y \in [0, X]$ and $[0, X] \cap \mathcal{X} = \emptyset$.

We first prove that $\mathcal{X}_0$ is Souslin-measurable. Since $O$ is open, for each element $(X, \epsilon)$ of $\text{FIN}^{[\omega]} \times (2^\omega)^{<\omega}$, $(X, \epsilon) \in O$ if and only if there exists $a \subseteq X$ and a pre-initial segment $\sigma$ of $\epsilon$ such that $[a] \times [\sigma] \subseteq O$. So $X_0$ can be written as

$$\mathcal{X}_0 = \{ X \in \text{FIN}^{[\omega]} : (\exists p \in \mathcal{P}_\omega)(\forall a \subseteq X)(\forall F \subseteq \omega \text{ finite})(\forall \sigma \in \prod_{i \in F} p(i))[a] \times [\sigma] \subseteq O \}.$$ 

Therefore, $\mathcal{X}_0$ is analytic, hence Souslin-measurable.

Lastly, we show that $\mathcal{X}_1$ is also analytic. Let $Y \in \text{FIN}^{[\omega]}$ and $p \in \mathcal{P}_\omega$. Assume $(Y) \times [p] \subseteq O$. As above, this implies that for every $\epsilon \in [p]$ there exists $a_\epsilon \subseteq Y$ and a pre-initial segment $\sigma_\epsilon$ of $\epsilon$ such that $[a_\epsilon] \times [\sigma_\epsilon] \subseteq O$. Thus $\{ [a_\epsilon] : \epsilon \in [p] \}$ is an open cover of $[p]$. Since $(2^\omega)^{<\omega}$ is compact, the closed subset $[p]$ is also compact. So $[p]$ has a finite subcover $\{ [\tau_1], \ldots, [\tau_k] \} \subseteq \{ [\sigma_\epsilon] : \epsilon \in [p] \}$. Correspondingly, there are $a_1, \ldots, a_k \subseteq Y$ such that $[a_i] \times [\tau_i] \subseteq O$ for every $i \leq k$. We can find $a \subseteq Y$ such that $a_i \subseteq a$ for every $i \leq k$. So $[a] \times [p] \subseteq O$. Thus we have proved that if $(Y) \times [p] \subseteq O$ then $Y$ has an initial segment $a$ such that $[a] \times [p] \subseteq O$. Note that the converse of this also holds. Therefore, for $X \in \text{FIN}^{[\omega]}$ and $p \in \mathcal{P}_\omega$, $[0, X] \times [p] \subseteq O$ if and only if every $Y \subseteq X$ has an initial segment in the set $\mathcal{F} = \{ a \subseteq X : [a] \times [p] \subseteq O \}$. By shrinking $\mathcal{F}$ to a front $\mathcal{F}'$ on $[0, X]$ and using Lemma 2.13 and Lemma 2.14, we have the following. For $X \in \text{FIN}^{[\omega]}$, $X \in \mathcal{X}_1$ if and only if

$$(\exists p \in \mathcal{P})(\exists S \subseteq \text{FIN}^{[\omega]})(\exists \mathcal{S} \subseteq \text{FIN}^{[\omega]})(S \text{ is uniform on } [0, X]) \land ((\forall a \in S)(\exists b \in \mathcal{F}')(b \subseteq a)),$$

and if and only if

$$(\exists p \in \mathcal{P})(\exists S \subseteq \text{FIN}^{[\omega]})(\exists \mathcal{S} \subseteq \text{FIN}^{[\omega]})(S \text{ is uniform on } [0, X]) \land ((\forall a \in S)(\exists b \subseteq X)((b \subseteq a) \land ([b] \times [p] \subseteq O))).$$

Thus $\mathcal{X}_1$ is also analytic.

\[ \square \]

### 2.2 Perfectly $\mathcal{U}$-Ramsey sets

Recall that we have fixed an arbitrary selective ultrafilter $\mathcal{U}$ on $\text{FIN}$ throughout Section 2. Unless otherwise stated, we are working with the metric topology on $\text{FIN}^{[\omega]}$, and the product topology on $(2^\omega)^{<\omega}$ and $\text{FIN}^{[\omega]} \times (2^\omega)^{<\omega}$. In this subsection, we generalize Theorem 2.15 to all Souslin-measurable subsets of $\text{FIN}^{[\omega]} \times (2^\omega)^{<\omega}$, thus proving that every selective ultrafilter on $\text{FIN}$ is localizing.

For a $\mathcal{U}$-tree $T$ and $x \in T$, let $T|x = \{ y \in T : y \subseteq x \text{ or } x \subseteq y \}$. Recall that $l_n(T)$ denotes the $n^{\text{th}}$ level of $T$ above its stem. Recall also that by $T' \leq T$ we mean $T' \subseteq T$ and $\text{st}(T') = \text{st}(T)$. This notion is extended as follows.

**Definition 2.16.** [20, Section 7.2]. For two $\mathcal{U}$-trees $T'$ and $T$, and $n \in \omega$, we write $T' \leq^n T$ if $T' \subseteq T$ and $l_n(T') = l_n(T)$.

**Definition 2.17.** [20, Section 7.2]. A sequence $(T_n)_{n<\omega}$ of $\mathcal{U}$-trees is a fusion sequence if $T_{n+1} \leq^n T_n$ for every $n \in \omega$. In this case, the set $T_\infty = \bigcap_{n<\omega} T_n$ is called the fusion of the sequence. Note that $T_\infty$ is also a $\mathcal{U}$-tree.

**Definition 2.18.** A subset $B$ of $\text{FIN}^{[\omega]} \times (2^\omega)^{<\omega}$ is perfectly $\mathcal{U}$-Ramsey if for every $\mathcal{U}$-tree $T$ and every $p \in \mathcal{P}_\omega$, there exist $T' \leq T$ and $p' \leq p$ such that $[T'] \times [p'] \subseteq B$ or $([T'] \times [p']) \cap B = \emptyset$. We say $B$ is perfectly $\mathcal{U}$-Ramsey null if the second alternative always holds.

By Lemma 2.8, the following lemma follows from Theorem 2.15.

**Lemma 2.19.** Every open subset of $\text{FIN}^{[\omega]} \times (2^\omega)^{<\omega}$ is perfectly $\mathcal{U}$-Ramsey.

Now we aim to show that the collection of perfectly $\mathcal{U}$-Ramsey sets is closed under the Souslin operation, through a series of lemmas. The following lemma is immediate from the definition of perfectly $\mathcal{U}$-Ramsey.
Lemma 2.20. A subset $B$ of $FIN^{[\omega]} \times (2^\omega)^\omega$ is perfectly $U$-Ramsey if and only if for every $U$-tree $T$ and $p \in \mathcal{P}_\omega$ and for every finite subset $F \subseteq \omega$, $n < \omega$, there exist $T' \leq^n T$ and $p' \leq^{F,n} p$ such that

$$\forall \sigma \in l(F,n,p) \quad [T'] \times [p'|\sigma] \subseteq B \text{ or } ([T'] \times [p'|\sigma]) \cap B = \emptyset.$$ 

Lemma 2.21. The set of all perfectly $U$-Ramsey null subsets of $FIN^{[\omega]} \times (2^\omega)^\omega$ is a $\sigma$-ideal.

Proof. Closure under subsets and finite union is clear. We show that it is closed under countable union. By Lemma 2.20 we notice that a set $B$ is perfectly $U$-Ramsey null if and only if for every $n, F, T, p$ given there exist $T' \leq^n T$ and $p' \leq^{F,n} p$ such that $([T'] \times [p']) \cap B = \emptyset$. Now suppose $Y_k$ is perfectly $U$-Ramsey null for every $k < \omega$. We prove that so is $\bigcup_{k<\omega} Y_k$. Given $T, p$ we construct fusion sequences $(T_k)_{k<\omega}$ and $(p_k)_{k<\omega}$ recursively.

Let $T_0 = T$, $p_0 = p$ and $F_k = \{0, \ldots, k\}$ for $k \in \omega$. At stage $k + 1$, enumerate $l_k(T_k)$ as $(x_j)_j < \omega$. Let $q_1 = p_k$. For each $j \in \omega$, as $Y_k$ is perfectly $U$-Ramsey null, there exist $S_j \subseteq T_k|_{x_j}$ and $q_j \leq^{F_k, k + j, j} q_1$ such that $([S_j] \times [q_j]) \cap Y_k = \emptyset$. Since $\bigcup_{j<\omega} F_k = \omega$, $(q_j)_j < \omega$ is a fusion sequence. Then let $T_{k+1} = \bigcup_{j<\omega} S_j$ and $p_{k+1} = \bigcap_{j<\omega} q_j$.

By construction, $T_{k+1}$ is a $U$-tree, $p_{k+1} \in \mathcal{P}_\omega$, $T_{k+1} \leq k, T_k, p_{k+1} \leq^{F_k, k, p_k}$ and $([T_k] \times [p_k]) \cap Y_k = \emptyset$. As $\bigcup_{k<\omega} F_k = \omega$, we can take the fusions $T_\infty = \bigcap_{k<\omega} T_k$ and $p_\infty = \bigcap_{k<\omega} p_k$. Then $[T_\infty] \times [p_\infty]$ is disjoint from $Y_k$ for each $k < \omega$, so it is disjoint from $\bigcup_{k<\omega} Y_k$. So $\bigcup_{k<\omega} Y_k$ is perfectly $U$-Ramsey null.

Lemma 2.22. The set of all perfectly $U$-Ramsey subsets of $FIN^{[\omega]} \times (2^\omega)^\omega$ is a $\sigma$-field.

Proof. Similarly we only show that it is closed under countable union. Suppose $Y_k$ is perfectly $U$-Ramsey for every $k < \omega$. We check that so is $\mathcal{Y} = \bigcup_{k<\omega} Y_k$. Given $T, p$ we construct fusion sequences $(T_k)_{k<\omega}$ and $(p_k)_{k<\omega}$ recursively.

Let $T_0 = T$, $p_0 = p$ and $F_k = \{0, \ldots, k\}$ for $k \in \omega$. At stage $k + 1$, let $l_k(T_k)$ as $(x_j)_j < \omega$. Let $q_1 = p_k$. For each $j \in \omega$, by Lemma 2.20, there exist $S_j \subseteq T_k|_{x_j}$ and $q_j \leq^{F_k, j, k+j, j} q_1$ such that for every $\sigma \in l(F_k, j, k+j)$ we have $([S_j] \times [q_j]) \cap Y_k = \emptyset$. Let $T_{k+1} = \bigcup_{j<\omega} S_j$ and $p_{k+1} = \bigcap_{j<\omega} q_j$. Then $T_{k+1}$ is a $U$-tree with $T_{k+1} \leq k, T_k$ and $p_{k+1} \in \mathcal{P}_\omega$.

Let $T_\infty = \bigcap_{k<\omega} T_k$ and $p_\infty = \bigcap_{k<\omega} p_k$. Then $[T_\infty] \leq T$, $p_\infty \leq p$ and for every $k < \omega$ and $x \in l_k(T_\infty)$,

$$\forall \sigma \in l(F_k, k + 1, p_\infty) \quad [T_\infty]|_x \times [p_\infty]|_\sigma \subseteq Y_k \text{ or } ([T_\infty]|_x \times [p_\infty]|_\sigma]) \cap Y_k = \emptyset.$$ 

Thus $\mathcal{Y} \cap ([T_\infty] \times [p_\infty])$ is open in $[T_\infty] \times [p_\infty]$ with the subspace topology. Then by Lemma 2.19, there exist $T' \leq T_\infty$ and $p' \leq p_\infty$ such that $[T'] \times [p'] \subseteq \mathcal{Y}$ or $([T'] \times [p']) \cap \mathcal{Y} = \emptyset$. In particular, $T' \leq T$ and $p' \leq p$ as required. Therefore $\mathcal{Y} = \bigcup_{k<\omega} Y_k$ is perfectly $U$-Ramsey.

We use combinatorial forcing ([16, 10]) to prove that the field of perfectly $U$-Ramsey subsets of $FIN^{[\omega]} \times (2^\omega)^\omega$ is closed under the Souslin operation. Recall that by $\sigma \in 2^{(\omega)\times(\omega)}$ we mean a function $\sigma : F \rightarrow 2^{<\omega}$ where $F \subseteq \omega$ is finite.

Definition 2.23. Consider an arbitrary $U$-tree $T$, $p \in \mathcal{P}_\omega$, $X \subseteq FIN^{[\omega]} \times (2^\omega)^\omega$ and $(a, \sigma) \in FIN^{[\omega] \times 2^{(\omega)\times(\omega)}}$. We say $(T, p) X$-accepts $(a, \sigma)$ if $[T|a] \times [p|\sigma] \subseteq X$; $(T, p)$ $X$-rejects $(a, \sigma)$ if $[T|a] \times [p|\sigma] \neq X$ and there does not exist $T' \leq T|a$ and $p' \leq p|\sigma$ such that $(T', p') X$-accepts $(a, \sigma)$; $(T, p) X$-decides $(a, \sigma)$ if it either $X$-accepts or $X$-rejects $(a, \sigma)$.

The following lemma is immediate from the definition.

Lemma 2.24. Consider an arbitrary $U$-tree $T$, $p \in \mathcal{P}_\omega$, $X \subseteq FIN^{[\omega]} \times (2^\omega)^\omega$ and $(a, \sigma) \in FIN^{[\omega] \times 2^{(\omega)\times(\omega)}}$.

1. $(T, p) X$-accepts every $(a, \sigma)$ such that $[T|a] \times [p|\sigma] = \emptyset$.
2. If $(T, p) X$-decides $(a, \sigma)$ and $[T'] \times [p'] \subseteq [T] \times [p]$, then $(T', p') X$-decides $(a, \sigma)$.
3. If $a \in T$ and $\sigma(n) \in p(n)$ for every $n \in \text{dom}(\sigma)$, then there exist $T' \leq T|a$ and $p' \leq p|\sigma$ such that $(T', p') X$-decides $(a, \sigma)$.
(4) If \((T,p)\) \(X\)-decides \((a,σ)\), then \((T,p)\) \(X\)-decides every element of the form \((a,σ')\) in the same way, where \(σ'(j)∈pσ(j)\) for every \(j∈\text{dom}(σ')\).

**Lemma 2.25.** Given a Souslin scheme \( (X_κ : s ∈ ω^ω) \) of subsets of \( \text{FIN}^{ω^ω} \times (2^ω)^ω \), a \( U \)-tree \( T \) and \( p ∈ P \), there exist \( T_{∞} \leq T \), \( p_{∞} ≤ p \) and an enumeration \( (b_n)_{n<ω} \) of \( \{ b ∈ T_∞ : \sigma(T) ⊆ b \} \) such that for every \( s ∈ ω^ω \) and every \( k ≥ max(s) \), if \( τ ∈ 2^{(ω^ω)×(ω^ω)} \) and \( |τ(j)| ≥ k \) for every \( j ∈ \text{dom}(τ) \), then \( T_∞, p_{∞} \) \( X_s \)-decides \((b_κ, τ)\).

**Proof.** Let \( \{ b ∈ T : \sigma(T) ⊆ b \} \) be enumerated as \((b_n)_{n<ω} \) such that \( b_m \not∈ b_n \) when \( n < m \). In particular, \( b_0 = \sigma(T) \). We recursively shrink \( T \) and \( p \) “cone by cone”, starting from \( T_{−1} = T \) and \( p_{−1} = p \). At stage \( k + 1 \), let \( F_{k + 1} = \{ 0, 1, \ldots, k + 1 \} \). Let \( l(F_{k + 1}, k + 1, pk) × \{ 0, 1, \ldots, k + 1 \} \) be enumerated as \((τ_i, s_i)_i<ω \) such \( m \) is a finite number. Let \( q_{−1} = pk \). By Lemma 2.24 (3), for every \( l ≤ m \) there exists \( S_l ⊆ T_{k+1} | b_{k+1} \) and \( q_l \leq F_{k+1,k+1} q_{l−1} \) with \( q_l \cdot τ_i ⊆ q_{l−1} \cdot τ_i \) such that \((S_l, q_l) X_{s_l} \) decides \((b_{k+1}, τ) \). Let \( p_{k+1} = q_m \), then \( p_{k+1} ≤ F_{k+1,k+1} p_k \) and \( p_{k+1} | τ_i ≤ q_i | τ_i \) for every \( l ≤ m \). Let \( T_{k+1} = (T_k \setminus T_k | b_{k+1}) \cup \bigcap_{l≤m} S_l \). Note that \( T_{k+1} \) is a \( U \)-tree. Then \((T_{k+1}, p_{k+1}) X_{s_{k+1}} \) decides \((b_{k+1}, τ) \) for every \( τ ∈ l(F_{k+1}, k + 1, pk) \) and every \( s ∈ \{ 0, \ldots, k + 1 \} ^ω \). Re-enumerate \( T_{k+1} \) as \((b_n)_{n<ω} \) such that \( b_n \not∈ b_n \) when \( n < m \).

Let \( T_∞ = \bigcap_{k<ω} T_k \) and \( p_{∞} = \bigcap_{k<ω} p_k \). Since \( \bigcap_{k<ω} F_k = ω \) and \( p_{k+1} ≤ F_{k+1,k+1} pk \) by construction, \((p_k)_{k<ω} \) is a fusion sequence and \( p_{∞} ∈ P \). We check that \( T_∞ \) is a \( U \)-tree. Let \( b ∈ T_∞ \). There exists \( n \) such that \( b \) is enumerated as \( b_{n+1} \) since stage \( n \). Then, \( T_{n+1} \) is a \( U \)-tree, \( \{ x ∈ \text{FIN} : b ⊆ b \cup \{ x \} \in T_{n+1} \} = \{ x ∈ \text{FIN} : b \cup \{ x \} \in T_{n+1} \} \in U \).

It remains to check that \( T_∞ \) and \( p_{∞} \) satisfy the theorem. Suppose \( n ∈ ω \) and \( s ∈ \{ 0, \ldots, n \} ^ω \). Let \( b ∈ T_∞ \) and \( τ ∈ 2^{(ω^ω)×(ω^ω)} \) with \( T_∞ | b \times p_{∞} | τ \neq 0 \) be given. Assume further that \( b \) is enumerated as \( b_k \) in \( T_∞ \) for some \( k ≥ n \), and \( |τ(j)| ≥ k \) for every \( j ∈ \text{dom}(τ) \). Let us check that \((T_∞, p_{∞}) X_s \)-decides \((b, τ) \). We can find \( τ' ∈ l(F_k, k, pk) = l(F_k, k, p_∞) \) such that \( |τ(j)| ≥ k_0 | τ(j)′ \) for every \( j ∈ \text{dom}(τ) \). Then \((T_k | p_k) X_s \) decides \((b_k, τ) \). By Lemma 2.24 (4), \((T_k | p_k) X_s \) also \( X_s \)-decides \((b_k, τ) \). Hence by Lemma 2.24 (2), \( [T_∞] \times [p_∞] X_s \)-decides \((b_k, τ) \) as well. On the other hand, if \( T_∞ | b \times p_{∞} | τ = 0 \), then by Lemma 2.24 (1), \( [T_∞] \times [p_∞] X_s \)-accepts \((b, τ) \).

**Theorem 2.26.** The field of perfectly \( U \)-Ramsey subsets of \( \text{FIN}^{ω^ω} \times (2^ω)^ω \) is closed under the Souslin operation.

**Proof.** Let \( (Y_s : s ∈ ω^ω) \) be a Souslin scheme of perfectly \( U \)-Ramsey sets. Assume without loss of generality that \( Y_λ ⊆ Y_λ \) whenever \( s ⊆ t \). Let \( \mathcal{Y} = \bigcup_{f∈ω} \bigcap_{n<ω} Y_f|n \). For a set \( \mathcal{X} ⊆ \text{FIN}^{ω^ω} \times (2^ω)^ω \), let \( \mathcal{X}^c \) denote its complement \( \text{FIN}^{ω^ω} \times (2^ω)^ω \setminus \mathcal{X} \). Given \( T, p \), we aim to show that there exist \( T' ≤ T \) and \( p' ≤ p \) such that \((T' \setminus T) \times [p' \setminus p] \not⊆ \mathcal{Y} \). By Lemma 2.25, we can find \( T_{∞} ≤ T \) and \( p_{∞} ≤ p \) where the elements in \( T_∞ \) above the stem are enumerated as \((b_n)_{n<ω} \) with \( b_m \not∈ b_n \) when \( n < m \) such that for every \( s ∈ ω^ω \) and every \( k ≥ max(s) \), if \( τ ∈ 2^{(ω^ω)×(ω^ω)} \) and \( |τ(j)| ≥ k \) for every \( j ∈ \text{dom}(τ) \), then \( T_∞ \times [p_∞] (Y_s^c) \)-decides \((b_k, τ) \). For each \( s ∈ ω^ω \), let \( O_s = \{ (b, τ) : T_∞ | b \times [p_∞] | τ = 0 \} \). We define an envelope for each \( Y_s^c \) by

\[
Φ(Y_s^c) = (T_∞ \times [p_∞] \cap Y_s) \setminus \bigcup_{(b, τ) ∈ O_s} (T_∞ | b \times [p_∞] | τ).
\]

Since \( T_∞ \times [p_∞] \), \( Y_s \) and \( T_∞ | b \times [p_∞] | τ \) are perfectly \( U \)-Ramsey, \( Φ(Y_s^c) \) is also perfectly \( U \)-Ramsey by Lemma 2.22. Clearly, \( T_∞ \times [p_∞] \cap Y_s^c \subseteq Φ(Y_s^c) \subseteq T_∞ \times [p_∞] \cap Y_s \).

We say a set \( B \) is perfectly \( U \)-Ramsey null inside \( T_∞ \times [p_∞] \) if for every \( U \)-tree \( S \) and \( q ∈ P_ω \) with \( [S] \times [q] ⊆ T_∞ \times [p_∞] \) there exist \( S' ≤ S \) and \( q' ≤ q \) such that \([S'] \times [q'] ⊆ B \).

**Claim 2.26.1.** For \( s ∈ ω^ω \), if \( B \subseteq Φ(Y_s^c) \setminus Y_s^c \) is perfectly \( U \)-Ramsey, then \( B \) is perfectly \( U \)-Ramsey null inside \( T_∞ \times [p_∞] \).

Suppose \( B ∈ Φ(Y_s^c) \setminus Y_s^c \) is perfectly \( U \)-Ramsey. Given \( [S] \times [q] ⊆ T_∞ \times [p_∞] \), we can find \( S' ≤ S \) and \( q' ≤ q \) such that \([S'] \times [q'] ⊆ B \) or \( B^c \). Assuming \([S'] \times [q'] ⊆ B \), we aim for a contradiction. Since \([S'] \times [q'] ⊆ T_∞ \times [p_∞] \), we can pick \( k ≥ max(s) \), \( b_k \) in \( S' \) and \( τ \) such that, for each \( j ∈ \text{dom}(τ) \), \( τ(j) \in q'(j) \) and \( |τ(j)| ≥ k \). Since we chose \( T_∞ \) and \( p_∞ \) to satisfy the conclusion of Lemma 2.25, \( (T_∞, p_∞) (Y_s^c) \)-decides \((b_k, τ) \). By Lemma 2.24(2), \((S', q') \) also \((Y_s^c)^c \)-decides \((b_k, τ) \). But \([S'] \times [q'] ⊆ B \) and \( B \) is disjoint
from \(Y^*\), we must have that \((S', q') (Y^*)^c\)-accepts \((b_k, \tau)\). Thus \((T_{\infty}, p_{\infty})\) must also \((Y^*)^c\)-accept \((b_k, \tau)\). Hence \((b_k, \tau) \in O_1\) and \([S'] \quad \| q' \| \subseteq \bigcup_{(b_k, \tau) \in O_1} [T_{\infty}] | \tau | \times [p_{\infty}] | \tau | \). This contradicts the assumption that \([S'] \quad \| q' \| \subseteq B\) which is disjoint from \(\bigcup_{(b_k, \tau) \in O_1} [T_{\infty}] | \tau | \times [p_{\infty}] | \tau | \). Therefore \([S'] \quad \| q' \| \subseteq B^c\). This finishes the proof of Claim 2.26.1.

For \(s \in \omega^{<\omega}\), let \(M_s = \Phi(Y^*_s) \cup \bigcup_{n < \omega} \Phi(Y^*_{s-n})\). Note that \(Y^*_s = \bigcup_{n < \omega} Y^*_{s-n}\) and \(\Phi(Y^*_{s-n}) \supseteq [T_{\infty}] | \tau | \times [p_{\infty}] \cap Y^*_{s-n}\). So \([T_{\infty}] | \tau | \times [p_{\infty}] \cap Y^*_s \subseteq \bigcup_{n < \omega} \Phi(Y^*_{s-n})\). Hence \(M_s \subseteq \Phi(Y^*_s) \cup \bigcup_{n < \omega} \Phi(Y^*_{s-n})\) since \(\Phi(Y^*_s) \subseteq [T_{\infty}] | \tau | \times [p_{\infty}] \cap Y^*_s\). As \(M_s\) is perfectly \(U\)-Ramsey, by Claim 2.26.1 \(M_s\) is perfectly \(U\)-Ramsey null inside \([T_{\infty}] \times [p_{\infty}]\). The set of all perfectly \(U\)-Ramsey null sets forms a \(\sigma\)-ideal (Lemma 2.21), so similarly \(\bigcup_{s < \omega^{<\omega}} M_s\) is also perfectly \(U\)-Ramsey null inside \([T_{\infty}] \times [p_{\infty}]\). Hence there exist \(T' \leq T_{\infty}\) and \(p' \leq p_{\infty}\) such that \([T'] \times [p'] \cap \bigcup_{s < \omega^{<\omega}} M_s \subseteq \emptyset\).

Claim 2.26.2. \(((T') \times [p']) \cap \Phi(Y^*_0) = ((T') \times [p']) \cap Y^*_0\).

The left inclusion is clear since \(\Phi(Y^*_0) \supseteq Y^*_0\). To prove the right inclusion, pick \(x \in (T') \times [p'] \cap \Phi(Y^*_0)\). We obtain \([T'] \times [p'] \cap M_0 = \emptyset\), so there exists \(n_0\) such that \(x \in \Phi(Y^*_{n_0})\). \([T'] \times [p'] \cap M_{n_0} = \emptyset\), so there exists \(n_1\) such that \(x \in \Phi(Y^*_{n_0,n_1})\). By repeating this process we find \(f = (n_k)_{k < \omega}\) such that \(x \in \Phi(Y^*_{f,k}) \subseteq Y^*_{f,k}\) for all \(k < \omega\). So \(x \in \bigcap_{k < \omega} Y^*_{f,k} \subseteq \emptyset = Y^*_0\). This finishes the proof of Claim 2.26.2.

Since \([T'] \times [p'] \cap \Phi(Y^*_0)\) is perfectly \(U\)-Ramsey, there exist \(T'' \leq T'\) and \(p'' \leq p'\) such that \([T''] \times [p''] \subseteq ((T') \times [p']) \cap \Phi(Y^*_0)\) or \(((T') \times [p']) \cap \Phi(Y^*_0))^c\). By Claim 2.26.2, \(((T') \times [p']) \cap \Phi(Y^*_0) = ((T') \times [p']) \cap Y^*_0 = ((T') \times [p']) \cap \emptyset = \emptyset\). Therefore \(Y\) is perfectly \(U\)-Ramsey. Thus we have proved that the field of perfectly \(U\)-Ramsey subsets of \(FIN^{<\omega} \times (2^\omega)^\omega\) is closed under the Souslin operation.

Corollary 2.27. Every Souslin-measurable subset of \(FIN^{<\omega} \times (2^\omega)^\omega\) is perfectly \(U\)-Ramsey.

Thus, we have proved that every selective ultrafilter on \(FIN\) is localizing. Therefore, as promised in the introduction, we see that a selective ultrafilter localizes the Parametrised Milliken Theorem 0.1 as follows.

Theorem 0.2 (Local Parametrized Milliken Theorem). Let \(U\) be a selective ultrafilter on \(FIN\). For every finite Souslin-measurable colouring of \(FIN^{<\omega} \times (2^\omega)^\omega\), there exist \([X] \in U\) and a sequence \((P_i)_{i < \omega}\) of nonempty perfect subsets of \(2^\omega\) such that \([X, X] \cap \bigcap_{i < \omega} P_i\) is monochromatic.

3 Selectivity preserved under Sacks forcings

Let \(U\) be an arbitrary selective ultrafilter on \(FIN\). We fix an arbitrary infinite cardinal \(\kappa\), and prove that \(U\) is preserved after adding \(\kappa\) Sacks reals using countable-support side-by-side Sacks forcing \(P_\kappa\). Recall that \(P_\kappa\) is the set of all functions \(p : \kappa \rightarrow \mathbb{P}\) with countable support, ordered by \(p \leq q\) if \(p(\alpha) \subseteq q(\alpha)\) for every \(\alpha < \kappa\). Forcing with \(P_\kappa\) adds a Sacks real for each \(\alpha \in \kappa\). From now on in this section, \(\models\) is used to denote the forcing relation with respect to \(P_\kappa\).

Lemma 3.1. [1, Lem. 1.9]. Suppose \(p \in P_\kappa, n \in \omega\) and that \(F \subseteq \kappa\) is finite. If \(q \leq p\) then there exists \(\sigma \in l(F, n, p)\) such that \(q \models p|\sigma\) are compatible.

Corollary 3.2. [1, Cor. 1.10]. Suppose \(p \in P_\kappa, n \in \omega\) and that \(F \subseteq \kappa\) is finite. If \(p \models \alpha \in V\) then there exists \(q \leq F, n\) \(p\) such that for each \(\sigma \in l(F, n, q)\) there exists \(a_\sigma \in V\) such that \(q|\sigma \models \alpha = a_\sigma\).

If \(q\) and \(\alpha\) as in Corollary 3.2. Then we say \(q\) determines \(\alpha\) relative to \((F, n)\). We say \(q\) determines \(\alpha\) if there exist \(F, n\) such that \(q\) determines \(\alpha\) relative to \((F, n)\). For \(q \models \alpha \in FIN \rightarrow V\), we say \(q\) determines \(\alpha\) if \(q\) determines \(\alpha(x)\) for every \(x \in FIN\).

As \(FIN\) is countable, we enumerate it as \(FIN = \{x_n : n \in \omega\}\).

Corollary 3.3. If \(p \in P_\kappa\) and \(p \models \alpha \in FIN \rightarrow 2\) then there exists \(q \leq p\) such that \(q\) determines \(\alpha\).

Proof. We construct a fusion sequence \((p_n : n < \omega)\) recursively, starting from \(p_0 = p\) and \(F_0 = \emptyset\). For \(n < \omega\), let \(F_{n+1} = F_n \cup \min(supp(p_n) \setminus F_n)\). By Corollary 3.2, there exists \(p_{n+1} \leq F_{n+1} p_n\) such that for each \(\sigma \in l(F_n, n, q)\), \(p_{n+1} \models \sigma\) determines \(\alpha(x_n)\). Then the fusion \(q = \bigcap_{n < \omega} p_n\) satisfies the statement. 

\[\square\]
Let $\mathcal{V}$ be a name with respect to $\mathcal{P}_\kappa$ for the upward closure of $\mathcal{U}$. Our aim is to prove that, forcing with $\mathcal{P}_\kappa$, $\mathcal{V}$ is a selective ultrafilter in the extension. It is straightforward to check the properties of $\mathcal{V}$ for being a selective ultrafilter except for the following.

Theorem 3.4. If $p \in \mathcal{P}_\kappa$ and $p \vdash \hat{\tau} \subseteq \text{FIN}$, then there exist $q \leq p$ and $[B] \in \mathcal{U}$ such that $q \vdash [B] \subseteq \hat{\tau}$ or $q \vdash [B] \cap \hat{\tau} = \emptyset$.

Proof. We abuse notation and use $\hat{\tau}$ to denote the characteristic function of the set, so $p \vdash \hat{\tau} : \text{FIN} \to 2$.

We will find $q \leq p$ and $[B] \in \mathcal{U}$ such that $q \vdash \hat{\tau}^* = [B]^\kappa$. By the proof of Corollary 3.3 we can construct a fusion sequence $(p_n : n \in \omega)$ with $(\mathcal{F}_n, n) : n \in \omega$ such that the fusion $p_\infty$ of the sequence satisfies the following condition.

$$\forall n < \omega \exists \sigma \in l(\mathcal{F}_n, n, p_\infty) \exists i_\sigma \in 2 \text{ such that } p_\infty \models \check{\tau}(x_n) = i_\sigma.$$ 

Note that $p_\infty \in \mathcal{P}_\kappa$, so $\text{supp}(p_\infty)$ is countable. We may assume without loss of generality that $\text{supp}(p_\infty) \subseteq \omega$. Then $[p'] \subseteq (2^\omega)^\omega$. We consider the following subset $\mathcal{F}$ of $\text{FIN} \times [p_\infty]$:

$$\mathcal{F} = \{(x_n, \epsilon) : (\exists \sigma \in l(F_n, n, p_\infty))(\forall i \in F_n)((\sigma(i) \in (\epsilon(i,j))_{j \in \omega}) \land (p_\infty \models \check{\tau}(x_n) = 0))\}.$$ 

Let $\mathcal{X} = \{(y_n)_{n=1}^\omega, \epsilon) \in \text{FIN}[^\omega] \times [p_\infty] : (y_1, \epsilon) \in \mathcal{F}\}$. Then $\mathcal{X}$ is open in $\text{FIN}[^\omega] \times [p_\infty]$ where $\text{FIN}[^\omega] \times [p_\infty] \subseteq \text{FIN}[^\omega] \times (2^\omega)^\omega$ has the subspace topology. Since $\mathcal{U}$ is localizing, there exists $[B] \in \mathcal{U}$ and $q \leq p_\infty$ such that $[\emptyset, B] \times [q] \subseteq \mathcal{X}$ or $([\emptyset, B] \times [q]) \cap \mathcal{X} = \emptyset$, hence $[B] \times [q] \subseteq \mathcal{F}$ or $(\mathcal{B} \times [q]) \cap \mathcal{F} = \emptyset$.

We check that $q$ and $[B]$ satisfy the theorem. Firstly, suppose $[B] \times [q] \subseteq \mathcal{F}$. We prove that if $x_n \in [B]$, then $q \vdash \check{\tau}(x_n) = 0$. Assuming there exists $r < q$ with $r \vdash \check{\tau}(x_n) = 1$, we aim for a contradiction. By Lemma 3.1, there exists $\sigma \in l(F_n, n, q)$ such that $r$ is compatible to $q/\sigma$. Let $\epsilon \in [q]$ be such that $\sigma$ is a pre-initial segment of $\epsilon$, that is, $\sigma(i) \in (\epsilon(i,j))_{j \in \omega}$ for every $i \in F_n$. Then, as $(x_n, \epsilon) \in \mathcal{F}$ and by the definition of $\mathcal{F}$, there exists a pre-initial segment $\sigma' \subseteq l(F_n, n, p_\infty)$ of $\epsilon$ such that $p_\infty \models \check{\tau}(x_n) = 0$. Since $q \leq p_\infty$, we have $\sigma'(i) \subseteq \sigma(i)$ for every $i \in F_n$, so $q/\sigma \leq p_\infty|\sigma'$. Therefore $q/\sigma \models \check{\tau}(x_n) = 0$. This contradicts that $r \Models \check{\tau}(x_n) = 1$ and $r, q/\sigma$ are compatible. If $[B] \times [q] \cap \mathcal{F} = \emptyset$, then we similarly have $q \Models \check{\tau}(x_n) = 1$ for every $x_n \in [B]$.

This completes the proof of Theorem 0.3 in the Introduction.

Theorem 0.3. Let $\kappa$ be an infinite cardinal, and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding $\kappa$ Sacks reals. Let $\mathcal{U}$ be a selective ultrafilter on $\text{FIN}$ in the ground model, and $\mathcal{V}$ a name for the upward closure $\{Y \subseteq \text{FIN} : \exists \check{X} \in \mathcal{U} \models \check{X} \subseteq Y\}$ of $\mathcal{U}$. Then $\vdash_{\mathcal{P}_\kappa} \mathcal{V}$ is a selective ultrafilter on $\text{FIN}$.

4 Selective ultrafilters are Ramsey

In [13], Mijares defined a notion of Ramsey ultrafilters in topological Ramsey spaces.

Definition 4.1. [13, Def. 3.2]. An ultrafilter $\mathcal{U}$ on $\text{FIN}$ is Ramsey if for $[X] \in \mathcal{U}, a \leq X$ and $n \in \omega$, and for every partition $\text{FIN}[|a| + n] = \mathcal{F}_0 \cup \mathcal{F}_1$, there exists $[Y] \in \mathcal{U}$ with $Y \in [a, X]$ such that the set $\{b \in \text{FIN}[|a| + n] : a \subseteq b \subseteq Y\}$ is contained in one of $\mathcal{F}_0, \mathcal{F}_1$.

Here we consider the following version used in [7], but we call it Nash-Williams instead of Ramsey. The change of name is because, although this property of Nash-Williams and Mijares’ property of Ramsey coincide in many topological Ramsey spaces, we do not know if there exists a space that distinguishes the two. For a family $\mathcal{F} \subseteq \text{FIN}[< \omega]$ and an infinite block-sequence $X \in \text{FIN}[\omega]$, let $\mathcal{F}[X] = \{a \in \mathcal{F} : a \leq X\}$.

Definition 4.2. [7, Def. 5.1]. An ultrafilter $\mathcal{U}$ on $\text{FIN}$ is Nash-Williams if for every Nash-Williams family $\mathcal{F} \subseteq \text{FIN}[< \omega]$ and every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ there exists $[X] \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{F}_i[X] = \emptyset$.

It is immediate that every Nash-Williams ultrafilter is Ramsey. In this section, we prove that every selective ultrafilter is Nash-Williams. Let $\mathcal{U}$ be an ultrafilter on $\text{FIN}$. Note that we do not assume $\mathcal{U}$ is selective here. Recall from Definition 2.6, the $\mathcal{U}$-topology on $\text{FIN}[\omega]$ is generated by sets of the form $[T]$ where $T$ is a $\mathcal{U}$-tree. It is a topology finer than the metric topology on $\text{FIN}[\omega]$. Recall from Definition 2.6...
that a set \( X \subseteq \text{FIN}^{[\infty]} \) is \( \mathcal{U} \)-Ramsey if every \( \mathcal{U} \)-tree \( T \) has a pure refinement \( T' \) such that \([T']\) is contained in or disjoint from \( X \).

First we note that every (metrically) open set in \( \text{FIN}^{[\infty]} \) is \( \mathcal{U} \)-Ramsey. The proofs of the following lemmas can be readily obtained by modifying the corresponding proofs found in [20] for the Ellentuck space. The proofs do not require \( \mathcal{U} \) to be selective.

**Definition 4.3.** [20, Def. 7.32]. A set \( G \subseteq \text{FIN}^{[<\infty]} \) is \( \mathcal{U} \)-open if for every \( a \in G \) there exists a \( \mathcal{U} \)-tree \( T \) with stem \( a \) such that \( \{ b \in T : a \sqsubseteq b \} \subseteq G \).

**Lemma 4.4.** [20, Lem. 7.33]. A set \( G \subseteq \text{FIN}^{[<\infty]} \) is \( \mathcal{U} \)-open if and only if for every \( s \in G \) the set \( G_s := \{ x \in \text{FIN} : s \cup x \in G \} \) is in \( \mathcal{U} \).

**Lemma 4.5.** [20, Lem. 7.38]. Every \( \mathcal{U} \)-open set is \( \mathcal{U} \)-Ramsey.

Now we show that every selective ultrafilter is Nash-Williams.

**Theorem 4.6.** If \( \mathcal{U} \) is a selective ultrafilter on \( \text{FIN} \), then \( \mathcal{U} \) is Nash-Williams.

**Proof.** Let \( F \subseteq \text{FIN}^{[<\infty]} \) be a Nash-Williams family and \( F = F_0 \sqcup F_1 \) be a partition. Then the set \( X := \bigcup_{a \in F_0} [a] \subseteq \text{FIN}^{[\infty]} \) is metrically open, which is therefore \( \mathcal{U} \)-Ramsey. So there exists a \( \mathcal{U} \)-tree \( T \leq^0 \text{FIN}^{[<\infty]} \) with stem \( \emptyset \) such that \([T] \subseteq X \) or \([T] \cap X = \emptyset \). Since \( \mathcal{U} \) is selective, by Lemma 2.8, we can find \([X] \in \mathcal{U} \) such that \([\emptyset, X] \subseteq [T] \).

If \([\emptyset, X] \subseteq X \), then every \( Y \leq X \) has an initial segment in \( F_0 \), and cannot have an initial segment in \( F_1 \) since \( F \) is Nash-Williams. Therefore \( F_1 \mid X = \emptyset \). Otherwise, \([\emptyset, X] \cap X = \emptyset \), hence \( F_0 \mid X = \emptyset \).

References


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