Pigeonhole, Tic-tac-toe and More

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Abstract

The Hales-Jewett theorem says that tic-tac-toe cannot end in a draw even if there is a huge number of players playing on a giant board, provided that the game is played in large-enough-dimension. This is a Ramsey-type theorem, showing that “complete disorder is impossible”.

We start from the pigeonhole principle and taste various delicious Ramsey-type theorems, including van der Waerden’s theorem, Ramsey’s theorem, the Hales-Jewett theorem, Hindman’s theorem and Milliken’s theorem. We also look at the relations among them, but no technical details are discussed.

“...complete disorder is impossible.” — Motzkin[1967]

Pigheaded principle If there are n people in the elevator and n+1 of the floor buttons are lit, then there exists a person who is pigheaded.

Example. Can you pick 51 integers from 1,2,...,100 such that none divides another? No.

Example. There are at least two of us having the same number of friends among people in this room.

Theorem 1 (Pigeonhole). For positive integers r,k there exists m such that for every r-colouring $f : m \rightarrow r$ then there exists $X \in m^{[k]}$ such that X is monochromatic. e.g. $m = r(k-1) + 1$.

Theorem 2 (infinite Pigeonhole). For every r and for every r-colouring $f : \mathbb{N} \rightarrow r$, there exists $X \in \mathbb{N}^{[\infty]}$ such that X is monochromatic.

Example. $r = 2, k = 3, m = 5$. For every $f : 5 \rightarrow 2$ there exists $X \in 5^{[3]}$ monochromatic:

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If we increase $m$ to 9, then a nicer pattern is guaranteed.

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Theorem 3 (Van der Waerden [1927]). For positive integers r,t there exists $M = W(r,t)$ such that for every r-colouring $f : M \rightarrow r$ there exists a monochromatic arithmetic progression of length t.

Theorem 4 (Ramsey [1930]). For positive integers r,n,k there exists m such that for every r-colouring $f : m^{[n]} \rightarrow r$ there exists $X \in m^{[k]}$ such that $X^{[n]}$ is monochromatic.

Complete disorder is impossible.
Example. \( n = 1 \) is the pigeonhole principle. When \( n = 2 \), we are \( r \)-colouring the edges of the complete graph with \( m \) vertices. \( R(3, 3) = 6 \) for \( r = 2, k = 3, n = 2 \).

**Theorem 5** (infinite Ramsey [1930]). For positive integers \( r, n \) and for every colouring \( f : \mathbb{N}^n \rightarrow r \) there exists \( X \in \mathbb{N}^{\infty} \) such that \( X^n \) is monochromatic.

Example. \( n = 1 \) is infinite pigeonhole. \( n = 2 \) is colouring an infinite complete graph.

1 **Tic-tac-toe**

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This is a draw. There must not be a draw if we have a large-dimensional board, e.g. an 8-dimensional board, even if certain diagonals do not count.

Let \( L \) be a finite alphabet, \( L^n \) be the set of all words of length \( n \) over the alphabet \( L \). This is our game board.

Example. \( L = \{1, 2, 3\}, n = 2 \),

\[ \{1, 2, 3\}^2 = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}. \]

Let \( v \) be a letter outside \( L \), a variable word \( w \) is a word over the alphabet \( L \cup \{v\} \) such that \( v \) occurs at least once in \( w \). For \( \lambda \in L \), the word \( w[\lambda] \) is obtained by replacing each occurrence of \( v \) with \( \lambda \). A combinatorial line in \( L^n \) is of the form \( \{w[\lambda] : \lambda \in L\} \) where \( w \) is a variable word of length \( n \).

Example. The variable word \( 11v2 \) gives the combinatorial line \( \{11, 12, 13\} \) which is the top row on the board; \( v2 \) gives \( \{12, 22, 32\} \) which is the middle column on the board; \( v3 \) gives \( \{11, 22, 33\} \) which is a diagonal on the board. Note that the other diagonal \( \{13, 22, 31\} \) is not a combinatorial line.

The combinatorial lines corresponds to rows, columns and some of the diagonals of the game board.

**Theorem 6** (Hales-Jewett [1963]). For every finite alphabet \( L \) and every positive integer \( r \) there exists \( n = HJ(r, |L|) \) such that for every \( r \)-colouring of the set \( W_L(n) \) of all \( L \)-words of length \( n \) there is a variable word \( w \) of length \( n \) such that the combinatorial line \( \{w[\lambda] : \lambda \in L\} \) is monochromatic.

Example. Clearly, \( HJ(2, 3) > 2 \). In fact, \( HJ(2, 3) \leq 18 \). \( 3^8 = 561 \).

So the theorem says that no matter how many players \( (r) \) there are and no matter how-many-in-a-row \( (|L|) \) is required to win, there must not be a draw when the game is played in large-enough-dimension.

We now show that the Hales-Jewett theorem is stronger than van der Waerden’s theorem. Recall that van der Waerden’s theorem says that for positive integers \( r, t \) there exists \( M = W(r, t) \) such that for every \( r \)-colouring \( f : \{1, \ldots, M\} \rightarrow r \) there exists a monochromatic arithmetic progression of length \( t \).

**Proof.** We show that \( M = tn \) works. Let \( r, t, f : \{1, \ldots, M\} \rightarrow r \) be given, let \( L = \{1, \ldots, t\} \) and \( n = HJ(r, t) \). Consider the function \( g : \{1, \ldots, t\}^n \rightarrow [M] \)

\[ x = (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i. \]

Then the following diagram commutes.

\[ \begin{array}{c|c|c|c|c}
2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & \end{array} \]

Note: summing up the digits gives the 3-in-a-row gives an AP of length 3.

\[ \sum x_i \leq M \] so \( M = tn \) is enough.

\[ ^1 \text{Hindman and Tessler [2014] proved } HJ(2, 3) = 4. \] Many thanks to Alex van den Brandhof who pointed it out to me.
Let $w$ be a variable word of length $n$. The corresponding combinatorial line \{w[1], \ldots, w[t]\} is of length $t$.

Note that, for $1 \leq i \leq t$, $g(w[i+1]) - g(w[i])$ is a constant, which equals the number of occurrence of $v$ in $w$. So $g$ maps \{w[1], \ldots, w[t]\} to an arithmetic progression of length $t$.

By HJ, there exists a monochromatic combinatorial line in $L^n$. So there exists a monochromatic arithmetic progression of length $t$ in \{1, \ldots, M\}.

The infinite versions of HJ were proved by Carlson-Simpson[1990] and [1987].