Parametrizing topological Ramsey spaces

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Abstract

We prove a general theorem which implies that essentially all infinite-dimensional Ramsey properties proven using topological Ramsey space theory can be parametrized by products of infinitely many perfect sets.

0 Introduction

Topological Ramsey spaces have been introduced in order to facilitate the study of infinite-dimensional Ramsey properties. They have been axiomatized in Carlson and Simpson [5], and Todorcevic [20]. A prototype example is the Ellentuck space $\mathbb{N}^{[\infty]}$, where Ellentuck [11] captured the Ramsey property of sets using the exponential topology, or the Ellentuck topology. The first parametrization of the infinite-dimensional Ramsey theorem is by Miller and Todorcevic [16]. In [16] Miller parametrized the Galvin-Prikry Theorem (See [12] by Galvin and Prikry), concerning classical Borel subsets of $\mathbb{N}^{[\infty]} \times \mathbb{R}$, using forcing and absoluteness. Pawlikowski [18] took an approach similar to Ellentuck’s in [11] to parametrize the Ellentuck theorem, concerning sets with the property of Baire with respect to the Ellentuck topology, thus strengthening the result in [16]. Mijares [15] proved the corresponding result in an abstract setting, parametrizing the abstract Ellentuck theorem with perfect sets, in topological Ramsey spaces. We conclude the introduction by noting that the original Ellentuck space allows the maximal parametrization which happens to be infinite products of finite sets of prescribed sizes as shown by Llopis-DiPrisco-Todorcevic ([6], [8]). A major open problem in this area asks if similar parametrization is possible for other topological Ramsey spaces. While we do not address this problem here, we expect that the ideas that we use here will be relevant for this problem as well.

In this paper, we parametrize the abstract Ellentuck theorem with countable sequences of perfect sets, for topological Ramsey spaces satisfying the condition $(L4)$, in addition to the axioms $(A1)$-$(A4)$ as defined in [20].

Definition 0.1. Let $\mathcal{R}$ be a topological Ramsey space. We say $\mathcal{R}$ satisfies $(L4)$ if the following holds.

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\end{itemize}
Let $p \in \mathcal{P}_\omega$, $A \in \mathcal{R}$ and $a \in \mathcal{AR}[\emptyset, A]$. Let \{\(O_b : b \in \mathcal{AR}_{a|+1}[a, A]\}\} be a family of open subsets of \([p]\). Then there exists $B \in [\text{depth}_{\omega}(a), A]$ and $q \leq p$ such that $O_b \cap [q]$ is constant on $b \in \mathcal{AR}_{a|+1}[a, B]$.

**Theorem 0.2** (Moderately-Abstract Parametrized Ellentuck Theorem). Let $\mathcal{R}$ be a topological Ramsey space satisfying (L4). For every finite Souslin-measurable colouring of $\mathcal{R} \times \mathbb{R}^N$ and for every $A \in \mathcal{R}, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [a, A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic.

The proof uses methods developed in [20, §9] and involves combinatorial forcing ([17, 12]) and the notion of envelope in abstract Baire theory.

The theorem is then applied to a particular hierarchy of topological Ramsey spaces – the High-dimensional Ellentuck spaces. These are spaces generalizing the Ellentuck space. In [10], Dobrinen found the Tukey type of the generic ultrafilter $G_2$ forced by $\mathcal{P}(\omega^2)/(\text{FIN}^{\leq \omega^3})$, which was a problem left open in [4]. Dobrinen [10] further constructed the topological Ramsey spaces $\mathcal{E}_k (k \geq 2)$, each being a dense subset of $(\text{FIN}^{\leq \omega^2})^+$, to show that the ultrafilters Tukey reducible to $G_k$ form a chain of length $k$, where $G_k$ are generic ultrafilters forced by $\mathcal{P}(\omega^k)/(\text{FIN}^{\leq \omega^k})$. It is also claimed that these forcings have complete combinatorics ([10]).

As the High-dimensional Ellentuck spaces generalizes the Ellentuck space, the following Parametrized High-dimensional Ellentuck Theorem extends [20, Thm. 4.43], which is the parametrization of the Ellentuck Theorem with countable sequences of perfect sets.

**Theorem 0.3** (Parametrized High-dimensional Ellentuck Theorem). For every finite Souslin-measurable colouring of $\mathcal{E}_k \times \mathbb{R}^N$ and for every $A \in \mathcal{E}_k, a \in \mathcal{AR}[\emptyset, A]$ and $p \in \mathcal{P}_\omega$ there exists $B \in [a, A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic.

Theorem 0.3 is in turn applied to obtain the preservation of the generic ultrafilter $G_k$ forced by $\mathcal{P}(\omega^k)/(\text{FIN}^{\leq \omega^k})$ under countable-support side-by-side Sacks forcing. Since $\mathcal{P}(\omega^k)/(\text{FIN}^{\leq \omega^k})$ is forcing equivalent to $(\mathcal{E}_k, \subseteq_{\text{FIN}^{\leq \omega^k}})$ ([10]), we limit our scope to $B_k := G_k \cap \mathcal{E}_k$.

**Theorem 0.4.** Let $k \in \omega, \kappa$ be an infinite cardinal and $\mathcal{P}_\kappa$ be countable-support side-by-side Sacks forcing adding $\kappa$ Sacks reals. Let $\mathcal{B}_k$ be a generic filter for $(\mathcal{E}_k, \subseteq_{\text{FIN}^{\leq \omega^k}})$, and $\dot{Y}$ a name for the upward closure $\{Y : (\exists X \in B_k)(X \leq Y)\}$ of $B_k$. Then $\Vdash_{\mathcal{P}_\kappa}(\dot{Y}$ is a Nash-Williams ultrafilter in $\mathcal{E}_k)$.

Baumgartner and Laver [3] proved in the Ellentuck space the preservation of selective ultrafilters under iterated Sacks forcing. It is well known that they are also preserved under side-by-side Sacks forcing. Similar results have been obtained for selective ultrafilters [22] in the Milliken space and Nash-Williams ultrafilters [21] in the spaces $\mathcal{R}_\kappa$ constructed by Dobrinen and Todorcevic [9]. In these cases, the ultrafilters are p-points, whereas the ultrafilters $G_k$ are non-p-points.

Section 1 contains preliminaries about topological Ramsey spaces, High-dimensional Ellentuck spaces, and Sacks forcing. In Section 2 we prove the Moderately-Abstract Parametrized Ellentuck Theorem 0.2. Section 3 is an application of Theorem 0.2, leading to the Parametrized High-dimensional Ellentuck Theorem 0.3. In Section 4 we discuss the properties of a generic filter...
\( B_k \) for \((\mathcal{E}_k, \subseteq_{\text{FIN}_k})\), show that it remains as a Nash-Williams ultrafilter after countable-support side-by-side Sacks forcing, proving Theorem 0.4.

1 Preliminaries

1.1 Topological Ramsey Spaces

In this subsection we recall relevant definitions and theorems about topological Ramsey spaces from [20].

Let \((R, \leq, r)\) be a triple such that \(R\) is a nonempty set, \(\leq\) is a quasi-ordering on \(R\) and \(r : R \times \omega \to AR\) is the restriction function which maps an element \(X \in R\) to the sequence \((r_n(X) = r(X, n))\) of finite approximations of \(X\).

The space \(R\) is equipped with the Ellentuck topology generated by basic open sets of the form
\[
[a, A] = \{B \in R : (B \leq A) \land (\exists n < \omega)(r_n(B) = a)\}
\]
for \(a \in AR\) and \(A \in R\). However, in this paper we consider the metric topology of \(R\), when \(R\) is considered as a subset of the Tychonoff cube \(AR^\mathbb{N}\), unless otherwise stated. The metric topology has basic open sets of the form
\[
[a] = \{A \in R : (\exists n < \omega)(r_n(A) = a)\}
\]
for \(a \in AR\). The Ellentuck topology is finer than the metric topology.

For \(a, b \in AR\), we say \(a\) is an initial segment of \(b\) and \(b\) is an end-extension of \(a\), and write \(a \sqsubseteq b\), if there exists \(A \in R\) and \(n \leq m < \omega\) such that \(a = r_n(A)\) and \(b = r_m(A)\); \(a \sqsubseteq b\) if \(a \sqsubseteq b\) as above and \(n < m\). We also write \(a \sqsubseteq A\) if \(a = r_n(A)\) for some \(n < \omega\). The length \(|a|\) of \(a\) is \(n\) if there exists \(A \in R\) such that \(a = r_n(A)\). If \(|a| < n\), then
\[\mathcal{AR}_n[a, A] = \{r_n(B) : (a \sqsubseteq B) \land (B \leq A)\}\]
Similarly,
\[\mathcal{AR}_n = \{r_n(X) : X \in R\}\]

Let
\[\mathcal{AR}[a, A] = \{b \in AR : (\exists n < \omega)(\exists X \in [a, A])r_n(X) = b\}, \text{ and}
\[|n, A| = [r_n(A), A].\]

For a quasi-ordering \(\leq_{\text{fin}}\) as in \((A2)\) below, we define
\[\text{depth}_A(a) = \min\{k : a \leq_{\text{fin}} r_k(A)\}\]
where we set \(\min \emptyset = \infty\).

The axioms for topological Ramsey spaces are as follows.

**Definition 1.1.** [20, §5.1]. We say \((R, \leq, r)\) is a topological Ramsey space if it is closed as a subset of \(\mathcal{AR}^\mathbb{N}\) and satisfies axioms \((A1)\) to \((A4)\).

\((A1)\)
1. \(r_0(A) = \emptyset\) for all \(A \in R\).
2. \(A \neq B\) implies \(r_n(A) \neq r_n(B)\) for some \(n\).
(3) \( r_n(A) = r_m(B) \) implies \( n = m \) and \( r_k(A) = r_k(B) \) for all \( k < n \).

(A2) There is a quasi-ordering \( \leq_{fin} \) on \( \mathcal{AR} \) such that

1. \( \{ a \in \mathcal{AR} : a \leq_{fin} b \} \) is finite for all \( b \in \mathcal{AR} \),
2. \( A \leq B \) if and only if \( (\forall n)(\exists m) r_n(A) \leq_{fin} r_m(B) \),
3. \( \forall a, b \in \mathcal{AR} [a \subseteq b \land b \leq_{fin} c \Rightarrow (\exists d \subseteq c)(a \leq_{fin} d)] \).

(A3) (1) If \( \text{depth}_B(a) < \infty \) then \([a, A] \neq \emptyset \) for all \( A \in [\text{depth}_B(a), B] \).
2. \( A \leq B \) and \([a, A] \neq \emptyset \) imply that there is \( A' \in [\text{depth}_B(a), B] \) such that \( \emptyset \neq [a, A'] \subseteq [a, A] \).
3. \( \text{depth}_B(a) < \infty \) and if \( O \subseteq \mathcal{AR}_{[a] +1} \), then there is \( A \in [\text{depth}_B(a), B] \) such that \( \mathcal{AR}_{[a] +1}[a, A] \subseteq O \) or \( \mathcal{AR}_{[a] +1}[a, A] \cap O = \emptyset \).

Definition 1.2. [20, Def. 5.1]. A subset \( \mathcal{X} \) of \( \mathcal{R} \) has the Ellentuck property of Baire if \( \mathcal{X} = O \triangle M \) for some Ellentuck open set \( O \subseteq \mathcal{R} \) and Ellentuck meagre set \( M \subseteq \mathcal{R} \).

Definition 1.3. [20, Def. 5.2]. A subset \( \mathcal{X} \) of \( \mathcal{R} \) is Ramsey if for every nonempty basic set \( [a, A] \) there is a \( B \in [a, A] \) such that \([a, B] \subseteq \mathcal{X} \) or \([a, B] \cap \mathcal{X} = \emptyset \). We say \( \mathcal{X} \) is Ramsey null if it is Ramsey and the second alternative always holds.

Theorem 1.4 (Abstract Ellentuck Theorem). [20, Thm. 5.4]. If \((\mathcal{R}, \leq, r)\) is a topological Ramsey space then every Ellentuck property of Baire subset of \( \mathcal{R} \) is Ramsey and every Ellentuck meagre subset is Ramsey null.

We will be using induction on the rank of barriers.

Definition 1.5. For a topological Ramsey space \( \mathcal{R} \) we consider a family \( \mathcal{F} \subseteq \mathcal{AR} \). We say \( \mathcal{F} \) is Nash-Williams if \( s \nsubseteq t \) for every distinct pair \( s, t \in \mathcal{F} \); \( \mathcal{F} \) is Sperner if \( s \nsubseteq t \) for every distinct pair \( s, t \in \mathcal{F} \).

Let \( a \in \mathcal{AR} \) and \( A \in \mathcal{R} \). We say \( \mathcal{F} \) is a barrier on \([a, A]\) if \( \mathcal{F} \) is Sperner and every \( X \in [a, A] \) has an initial segment in \( \mathcal{F} \). We say \( \mathcal{F} \) is a barrier on \( A \) if it is a barrier on \([\emptyset, A]\).

Definition 1.6 (Rank of barriers). [20, Def. 1.24]. Let \( \mathcal{R} \) be a topological Ramsey space, \( a \in \mathcal{AR} \) and \( A \in \mathcal{R} \). Let \( \mathcal{F} \) be a barrier on \([a, A]\). Consider \( T(\mathcal{F}) = \{ s \in \mathcal{AR} : (a \subseteq s \leq A) \land (\exists t \in \mathcal{F})(s \subseteq t) \} \) as a tree ordered by end-extension \( \subseteq \). We define a strictly decreasing map \( \rho_\mathcal{F} \) as follows.

\[
\rho_\mathcal{F} : T(\mathcal{F}) \to \text{Ord} \\
\quad s \mapsto \sup \{ \rho_\mathcal{F}(t) + 1 : (t \in T(\mathcal{F})) \land (s \subseteq t) \}.
\]

The rank of \( \mathcal{F} \) on \([a, A]\) is \( rk(\mathcal{F}) = \rho_\mathcal{F}(a) \).

We will also be using the technique of fusion in Ramsey spaces.

Definition 1.7. [20, Def. 4.25]. Let \( \mathcal{R} \) be a topological Ramsey space. A sequence \( (n_i, Y_i)_{i<\omega} \) of basic subsets of \( \mathcal{R} \) is a fusion sequence if \((n_i)_{i<\omega}\) is an unbounded increasing sequence of integers, and \( Y_{i+1} \in [n_i, Y_i] \) for all \( i < \omega \). The fusion \( \lim Y_i \) of the sequence is the unique element \( Y_\infty \in \mathcal{R} \) such that \( r_{n_i}(Y_\infty) = r_{n_i}(Y_i) \) for all \( i < \omega \).
1.2 High-dimensional Ellentuck Spaces

In [10] Dobrinen presented a hierarchy \((E_k)_{k<\omega}\) of topological Ramsey spaces. The Ellentuck space naturally fits in the hierarchy as \(E_1\). The members of \(E_{k+1}\) look like \(\omega\) many copies of the members of \(E_k\). Therefore, \(E_k\) are called High-dimensional Ellentuck spaces. Let us recall the structure of \(E_k\) from [10], for an arbitrarily fixed integer \(k \geq 2\).

Let \(\omega^{jk}\) (resp. \(\omega^{j<k}\)) be the set of non-decreasing sequences of members of \(\omega\) with length \(k\) (resp. \(\leq k\)). The set \(\omega^{j<k}\) is equipped with a well-ordering \(<\) of order-type \(\omega\).

**Definition 1.8.** For \(\vec{j} = (j_0, \ldots, j_{p-1})\) and \(\vec{l} = (l_0, \ldots, l_{q-1})\) in \(\omega^{j<k}\), we write \(\vec{j} \leq_{\text{lex}} \vec{l}\) if \(\vec{j} \subseteq \vec{l}\) or \(\vec{j}\) precedes \(\vec{l}\) in the lexicographical ordering; \(\vec{j} <_{\text{lex}} \vec{l}\) if either \(\vec{j} \leq_{\text{lex}} \vec{l}\) and \(\vec{j} \neq \vec{l}\), or \(\vec{j}\) precedes \(\vec{l}\) in the lexicographical ordering; \(\vec{j} < \vec{l}\) if either \(j_{p-1} < l_{q-1}\) or \(j_{p-1} = l_{q-1}\) and \(\vec{j} <_{\text{lex}} \vec{l}\).

Note that we use \(\subseteq\) as initial segments between sequences here. The first few elements in \(\omega^{j<k}, <\) are as follows:

\[
(0), (0,0), (0,1), (1), (1,1), (0,2), (1,2), (2, 2), (0,3), \ldots
\]

The definitions below from [1] are equivalent to their original forms in [10].

**Definition 1.9 ([1]).** A function \(\hat{X} : \omega^{j<k} \to \omega^{j<k}\) is an \(E_k\)-tree if it preserves the well-ordering \(<\) and initial segments \(\subseteq\) on \(\omega^{j<k}\). For an \(E_k\)-tree \(\hat{X}\), let \(X\) be the restriction of \(\hat{X}\) to \(\omega^{j<k}\). We may identify \(X\) with its range and enumerate its elements \(<\)-increasingly.

**Definition 1.10 (Space \((E_k, \leq, r)\)).** Let \(E_k\) be the set of all \(X\) such that \(\hat{X}\) is an \(E_k\)-tree. For \(X, Y \in E_k\), \(Y \leq X\) if (the range of) \(Y\) is a subset of (the range of) \(X\). If \(X \in E_k\) is enumerated \(<\)-increasingly as \(X = (v_i)_{i<\omega}\) then \(X \subseteq \omega^{j<k} \subseteq [\omega]^k\) and we write

\[
[X] = \{\max(v_i) : i < \omega\} \subseteq \omega;
\]

for \(n < \omega\), let

\[
r_n(X) = \{v_0, \ldots, v_n\}.
\]

We write \(A\mathcal{E}_n^{E_k}\) for the set \(A\mathcal{R}_n\) when \(R = E_k\).

Note that \(\omega^{j<k}\) is the greatest element in \(E_k\). Similar to \([X]\), for \(S \subseteq [\omega]^k\), let

\[
[S] = \{\max(\zeta) : \zeta \in S\}.
\]

**Definition 1.11.** For \(\vec{j} \in \omega^{j<k}\), let \(|\vec{j}|\) denote the length of the sequence \(\vec{j}\). For \(l < |\vec{j}|\), we defined the map \(\pi_l\) as follows so that it projects members of \(\omega^{j<k}\) to their initial segments of length \(l\): if \(\vec{j} = (j_m)_{m<|\vec{j}|}\), then let

\[
\pi_l(\vec{j}) = (j_m)_{m < l}.
\]

Enumerate \(\omega^{j<k}\) \(<\)-increasingly as \((i_n)_{n<\omega}\). For \(l < k\), we define the set \(N^k_l \subseteq \omega\) by setting \(n \in N^k_l\) if and only if

\[
l = \max\{l' : (\exists m < n)(\pi_{l'}(\vec{t}_{n+1} \subseteq \vec{t}_m))\}.
\]
Since the elements of $\mathcal{E}_k$ preserves $\prec$ and $\sqsubset$, $n \in N^k_i$ if and only if for an $n$th approximation $a$ in $\mathcal{A}^E_{n^k}$, every 1-extension of $a$ has an extra branch, branching off from the $l$th level in the tree $a$. (We draw trees upwards, with the root $\emptyset$ at the bottom, being level 0.)

In Section 3 we will use the upper triangular representation of $\omega^{l^2}$ introduced in [1]. The idea of this representation is visualised through the Figures 1, 2 and Tables 1, 2.

Figure 1: $\mathcal{E}_2$-tree $\omega^{l^2}$

<table>
<thead>
<tr>
<th>(0,0)</th>
<th>(0,1)</th>
<th>(0,2)</th>
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<td>(4,4)</td>
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Table 1: Upper triangular representation of $\omega^{l^2}$

Figure 2: An example $\hat{X}$ of $\mathcal{E}_2$-tree, where

$X = \{(2,4), (2,6), (6,6), (2,8), (6,8), (9,9), (2,10), (6,11), (9,12), (14,15), \ldots \}$

1.3 Sacks Forcings

We equip $2^\omega$ and $(2^\omega)^\omega$ with the product topology. We use $2^\omega$ interchangeably with $\mathbb{R}$, and $(2^\omega)^\omega$ interchangeably with $\mathbb{R}^{\mathbb{N}}$. As usual, "$|\cdot|$", "$\sqsubseteq$" and "$|$" respectively denote length of the sequence, initial segment and restriction to an
Table 2: Upper triangular representation of $X$, with members of $X$ shaded, where

$$X = \{(2, 4), (2, 6), (6, 6), (2, 8), (6, 8), (9, 9), (2, 10), (6, 11), (9, 12), (14, 15), \ldots \}$$

initial 01-sequence of certain length on the set $2^{<\omega}$ of all finite 01-sequences. Two
finite 01-sequences are comparable if one is an initial segment of the other, and incomparable otherwise.

**Definition 1.12.** [2]. We say a nonempty set $p \subseteq 2^{<\omega}$ is a tree if it is closed
under initial segments. A tree $p$ is perfect if every $s \in p$ has end-extensions $t, u \in p$ that are incomparable. Note that every perfect tree is infinite.

For a perfect tree $p$, let $[p]$ be the set of all infinite branches of $p$:

$$[p] = \{ f \in 2^{\omega} : (\forall n \in \omega)(f \upharpoonright n \in p) \}.$$ 

Then $[p] \subseteq 2^{\omega}$ is a perfect set.

**Definition 1.13.** [19]. Sacks forcing $\mathcal{P}$ is the set of all perfect trees, with $p \leq q$
if $p \subseteq q$.

Note that $p \leq q$ if and only if $[p] \subseteq [q]$.

**Definition 1.14.** [2]. For $p \in \mathcal{P}$ and $s \in p$, let $p|s = \{ t \in p : ( t \subseteq s ) \lor ( s \subseteq t ) \}$.
The branching level of $s$ in $p$ is the number of branchings below $s$ in the tree $p$:

$$\{ |i < |s| : (\exists t \in p)((|t| > i) \land (t \upharpoonright i = s \upharpoonright i) \land (t \upharpoonright (i + 1) \neq s \upharpoonright (i + 1))) \}.$$ 

The $n$th branching level $l(n, p)$ of the tree $p$ is the set of all $s \in p \subseteq$-minimal
such that the branching level of $s$ in $p$ is $n$.

For $p, q \in \mathcal{P}$ and $n \in \omega$, $q \leq^n p$ if

$$q \subseteq p \text{ and } l(n, p) = l(n, q).$$ 

Note that $l(n, p) \subseteq p$ is a collection of nodes in $p$.

**Lemma 1.15** (Fusion 1). [2, Lem. 1.4]. Suppose $(p_k)_{k \in \omega} \subseteq \mathcal{P}$ and $(m_k)_{k \in \omega} \subseteq \omega$ is increasing and unbounded such that $p_{k+1} \leq^{m_k} p_k$ for all $k \in \omega$. Then $q = \bigcap_{k \in \omega} p_k \in \mathcal{P}$ and $q \leq^{m_k} p_k$ for all $k \in \omega$.

In such case, we call $(p_k)_{k \in \omega}$ a fusion sequence and $q$ the fusion of the sequence.
We have defined \( l(n, p) \) to be the \( n \)th branching level of \( p \). On the other hand, we let \( (p)_n \) be the \( n \)th level of \( p \), i.e.

\[
(p)_n = \{ s \in p : |s| = n \}.
\]

This symbol is used when we apply the Halpern-Läuchli theorem in Section 3.

Now we define countable-support side-by-side Sacks forcing.

**Definition 1.16.** [2]. Let \( \kappa \) be an infinite cardinal. Let \( \mathcal{P}_\kappa \) be the set of all sequences \( p = (p(i))_{i < \kappa} \) such that, for every \( i < \kappa \), \( p(i) \in \mathcal{P} \) and for all but countably many \( i < \kappa \), \( p(i) = 2^{<\omega} \). We say \( p(i) \) is the \( i \)th tree of \( p \). The support of \( p \) is \( \text{supp}(p) = \{ i < \kappa : p(i) \neq 2^{<\omega} \} \). So each \( p \in \mathcal{P}_\kappa \) has countable support.

For \( p \in \mathcal{P}_\kappa \), let \( [p] = \prod_{i < \kappa} [p(i)] \). For \( \varepsilon \in [p] \) and \( i < \kappa \), let \( \varepsilon(i) \) be the \( i \)th component in \( \varepsilon \), so \( \varepsilon(i) \in [p(i)] \). For \( p = (p(i))_{i < \kappa} \) and \( q = (q(i))_{i < \kappa} \) in \( \mathcal{P}_\kappa \), \( p \leq q \) if \( p(i) \subseteq q(i) \) for all \( i < \kappa \).

For a set \( S \), \( [S]^{<\omega} \) denotes the set of all finite subsets of \( S \).

**Definition 1.17.** Let \( \kappa \) be an infinite cardinal. Let \( F \in [\kappa]^{<\omega} \) and \( p \in \mathcal{P}_\kappa \). The set \( l(F, p) \) is defined as follows.

\[
l(F, p) = \prod_{i \in F} l(|F|, p(i)).
\]

For \( \sigma \in l(F, p) \) and \( i \in F \), let \( \sigma(i) \) denote the \( i \)th component of \( \sigma \), so \( \sigma(i) \in p(i) \). In such case we write \( \text{dom}(\sigma) \) for \( F \). If there exists \( n \in \omega \) such that \( F = \{ 0, 1, \ldots, n - 1 \} \), then we may write \( l(n, p) \) for \( l(F, p) \).

For \( p, q \in \mathcal{P}_\kappa \), we write \( q \leq^F p \) if \( q \leq p \) and \( q(i) \leq |F| \cdot p(i) \) for all \( i \in F \), equivalently,

\[
q \leq^F p \text{ if } q \leq p \text{ and } l(F, p) = l(F, q).
\]

For \( F, p, q \) as above and \( \sigma \in l(F, p) \), we write \( q \leq_\sigma p \) if \( q \leq p \) and \( \sigma(i) \in q(i) \) for every \( i \in F \). We also define \( p|\sigma \) as follows. For \( i < \kappa \),

\[
p|\sigma(i) = \begin{cases} p(i)|\sigma(i) & \text{if } i \in \text{dom}(\sigma); \\ p(i) & \text{otherwise}. \end{cases}
\]

Moreover, let \( \varepsilon \in [p] \) and \( \sigma \in l(F, p) \). We say \( \sigma \) is a pre-initial segment of \( \varepsilon \) and \( \varepsilon \) is a post-end-extension of \( \sigma \), and write \( \sigma \preceq^* \varepsilon \), if \( \sigma(i) \in \varepsilon(i) \) for every \( i \in \text{dom}(\sigma) \).

**Lemma 1.18** (Fusion 2). [2, Lem. 1.6]. Let \( \kappa \) be an infinite cardinal. Suppose \( (p_k)_{k \in \omega} \subseteq \mathcal{P}_\kappa \). Suppose also that \( (F_k)_{k \in \omega} \subseteq [\kappa]^{<\omega} \) is an unbounded \( \subseteq \)-increasing sequence with \( \bigcup_{k \in \omega} F_k \supseteq \bigcup (\text{supp}(p_k) : k \in \omega) \) and \( p_{k+1} \leq^F p_k \). Define \( q = (q(i))_{i < \kappa} \) where \( q(i) = \bigcap_{k \in \omega} p_k(i) \) for each \( i < \kappa \). Then \( q \in \mathcal{P}_\kappa \) and \( q \leq^F p_k \) for all \( k \in \omega \).

Recall that \( 2^\omega \) has the product topology. So it has basic open sets of the form

\[
[s] = \{ f \in 2^\omega : s \sqsubset f \} \text{ for } s \in 2^{<\omega}.
\]

Let \( \kappa \) be an infinite cardinal. The set \( (2^\kappa)^\kappa \) also has the product topology, with basic open sets of the form

\[
[\sigma] = \{ \varepsilon = (\varepsilon(i))_{i < \kappa} \in (2^\kappa)^\kappa : \sigma \preceq^* \varepsilon \},
\]

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where \( \sigma(i) \in 2^{<\omega} \) for every \( i \) in its domain, and \( \text{dom}(\sigma) \in [\kappa]^{<\omega} \). If \( \kappa = \omega \), we may think of such \( \sigma \) as an element of \( (2^{<\omega})^{<\omega} \), or \( 2^{(\omega \times (\omega \times \omega))} \). For \( p \in \mathcal{P}_n \), \([p]\) inherits the subspace topology from \( (2^{\omega})^{<\omega} \).

We keep in mind that \( \sqsubseteq \) denotes end-extension in different cases: We may use \( \sqsubseteq \) as the relation of end-extension on \( AR \times (AR \cup R) \). We may also use it to denote end-extension among nodes in a tree, e.g. \( 2^{\omega} \), \( \mathcal{W}_k \) and \( \omega^{\leq k} \), where the nodes are finite sequences.

In these cases, we write \( a \sqsubseteq b \) if \( a \sqsubseteq b \) and \( a \neq b \).

## 2 Parametrizing Moderate Ramsey spaces

In this section, we assume \( R \) is a topological Ramsey spaces satisfying (L4):

(L4) Let \( p \in \mathcal{P}_\omega \), \( A \in R \), and \( a \in AR[\emptyset, A] \). Let \( \{O_b : b \in AR[a]+1[a, A]\} \) be a family of open subsets of \([p]\). Then there exists \( B \in [\text{depth}_A(a), A] \) and \( q \leq p \) such that \( O_b \cap [q] \) is constant on \( b \in AR[a]+1[a, B] \).

### Lemma 2.1. [21, Lem. 2.1].

For \( p \in \mathcal{P}_\omega \) and \( O \subseteq [p] \) open, there exists \( q \leq p \) such that \( [q] \subseteq O \) or \([q] \cap O = \emptyset\).

### Lemma 2.2.

Suppose (L4) holds. Let \( p \in \mathcal{P}_\omega \), \( A \in R \), and \( a \in AR[\emptyset, A] \). Let \( n \in \omega \). Let \( F \) be a barrier on \([a, A]\) and \( O_b (b \in F) \) be a family of open subsets of \([p]\). Then there are \( B \in [\text{depth}_A(a), A], q \leq n \) and a clopen set \( G \subseteq [q] \) such that \( O_b \cap [q] = G \) for all \( b \in F \cap AR[a, B] \).

**Proof.** Induct on \( \text{rk}(F) \). The base case \( \text{rk}(F) = 0 \) is trivial, and that the set \( G \) is clopen follows from Lemma 2.1 applied to \([p]_\sigma\) for every \( \sigma \in l(n, p) \).

Suppose \( \text{rk}(F) > 0 \). We can write

\[
F = \bigcup_{b \in AR[a]+1[a, A]} F_b,
\]

where

\[
F_b = \{ c \in F : r_{a+1}(c) = b \}.
\]

Whenever \( a \sqsubseteq b \in AR[a, A] \) and \( F_b \neq \emptyset \), \( F_b \) is a barrier on \([b, A]\) of smaller rank than \( F \).

We build fusion sequences \( ([n_k, B_k])_{k<\omega} \) and \( (p_k)_{k<\omega} \) such that for \( k \geq 1 \):

1. \( B_0 = A \), \( p_0 = p \) and \( n_k = k + \text{depth}_A(a) \);

2. for each \( b \in F \cap AR[a, A] \) such that \( \text{depth}_{B_k}(b) = n_k \), we have that \( c \mapsto O_c \cap [p_{k+1}] \) is constant on \( F_b \cap AR[b, B_{k+1}] \);

3. \( p_{k+1} \leq n_{k+1} \) \( p_k, B_{k+1} \) is in \([n_k, B_k]\).

Suppose we have constructed \( B_k, p_k \). The set

\[
\{ b \in AR[a, A] : \text{depth}_{B_k}(b) = n_k \}
\]

is a subset of \( \{ b \in AR : b \leq \text{fin } r_{n_k}(B_k) \} \), so it is finite by (A2) (1). Enumerate it as \( b_0, \ldots, b_{l-1} \), and construct \( (Y_i)_{i \leq l} \) and \( (q_i)_{i \leq l} \) such that...
(i) \( Y_0 = B_k \), \( q_0 = p_k \);
(ii) \( Y_{i+1} \in [n_k, Y_i], q_{i+1} \leq n^k \)
(iii) \( c \mapsto O_c \cap [q_{i+1}] \) is constant on \( \mathcal{F}_b \cap \mathcal{AR}[b, Y_{i+1}] \).

Suppose we have constructed \( Y_i, q_i \) for some \( i < l \). If \( \mathcal{F}_b = \emptyset \), then let \( Y_{i+1} = Y_i \) and \( q_{i+1} = q_i \). Now assume \( \mathcal{F}_b \neq \emptyset \). Since \( \mathcal{F}_b \) is a barrier on \([b, Y_i]\) of smaller rank than \( \mathcal{F} \), by applying the induction hypothesis to \( b \), \( Y_i \) and \( q_i \) we find \( Y_{i+1} \in [n_k, Y_i], q_{i+1} \leq n^k \) \( q_i \) and \( G_{i+1} \subseteq [q_{i+1}] \) clopen such that
\[
O_c \cap [q_{i+1}] = G_{i+1} \quad \text{for all} \quad c \in \mathcal{F}_b \cap \mathcal{AR}[b, Y_{i+1}].
\]

This finishes the construction of \((Y_i)_{i < l}\) and \((q_i)_{i < l}\). Let \( B_{k+1} = Y_l \) and \( p_{k+1} = q_l \). This in turn finishes the construction of \((\{n_k, B_k\})\) and \((p_k)\). Then let \( B_{\infty}, \rho_\infty \) be the fusions of the sequences.

**Claim 2.2.1.** The map \( c \mapsto O_c \cap [p_\infty] \) restricted to \( \mathcal{F} \cap [a, B_\infty] \) depends only on \( r_{[a]+1}(c) \).

**Proof.** Suppose \( c, c' \in \mathcal{F} \cap \mathcal{AR}[a, B_\infty] \) such that \( r_{[a]+1}(c) = b = r_{[a]+1}(c') \). Let \( n_k = \text{depth}_{B_\infty}(b) \). So \( n_k = \text{depth}_{B_k}(b) \). Therefore
\[
c \mapsto O_c \cap [p_{k+1}] \quad \text{is constant on} \quad \mathcal{F}_b \cap \mathcal{AR}[b, B_{k+1}].
\]

Since \( c, c' \in \mathcal{F}_b \cap \mathcal{AR}[b, B_{k+1}] \) and \([p_\infty] \subseteq [p_{k+1}]\), the result follows. \( \square \)

Finally, apply \((L4)\) to find \( B \leq B_\infty \) and \( q \leq p_\infty \) such that \( c \mapsto O_c \cap [q] \) is constant on \( \mathcal{F} \cap \mathcal{AR}[a, b] \). \( \square \)

**Corollary 2.3.** Suppose \((L4)\) holds. Let \( A \in \mathcal{R} \) and \( a \in \mathcal{AR} \) with \( \text{depth}_4(A) < \infty \). Let \( \mathcal{F} \) be a barrier on \([a, A]\) and \( O_b(b \in \mathcal{F}) \) be a family of open subsets of \((2^{<\omega})^\omega\). Then for every \( p \in \mathcal{P}_\omega \) there exists \( q \leq p \), \( B \in [\text{depth}_4(A), A] \) such that either \([q] \subseteq O_b \) for every \( b \in \mathcal{F} \cap \mathcal{AR}[a, B] \) or \([q] \cap O_b = \emptyset \) for every \( b \in \mathcal{F} \cap \mathcal{AR}[a, B] \).

Towards proving the Moderately-Abstract Parametrized Ellentuck Theorem, we consider first the open subsets of \( \mathcal{R} \times \mathbb{R}^N \). We prove a corresponding result in Subsection 2.1, and generalize it to all Souslin-measurable subsets in Subsection 2.2.

### 2.1 Open Subsets of \( \mathcal{R} \times \mathbb{R}^N \)

In this subsection, we fix an open subset \( \mathcal{O} \) of \( \mathcal{R} \times \mathbb{R}^N \). We use combinatorial forcing to prove Theorem 2.10 below.

**Definition 2.4.** Let \( a \in \mathcal{AR}, A \in \mathcal{R}, p \in \mathcal{P}_\omega \) and \( \sigma \in 2^{(\omega)^\omega} \times (\omega)^\omega \). We say \((A, p)\) accepts \((a, \sigma)\) if \([a, A] \times [p|\sigma] \subseteq \mathcal{O}; (A, p)\) rejects \((a, \sigma)\) if \([a, A] \times [p|\sigma] \neq \emptyset \) and there does not exist \( q \leq \sigma \) and \( B \in [\text{depth}_4(a), A] \) such that \((B, q)\) accepts \((a, \sigma)\). We say \((A, p)\) decides \((a, \sigma)\) if it either accepts or rejects \((a, \sigma)\).

**Lemma 2.5.** Let \( a \in \mathcal{AR}, A \in \mathcal{R}, p \in \mathcal{P}_\omega \) and \( \sigma \in 2^{(\omega)^\omega} \times (\omega)^\omega \).
Proof. It is straightforward to see that (3) follows from (1) and (2), and that (4) follows directly from Definition 2.4. So we prove (1), (2), (5) and (6).

(1) $B \subseteq A$ and $q \leq p$ imply that $[a, B] \times [q|\sigma]$, which may be empty, is included in $[a, A] \times [p|\sigma]$ as a subset. So the result follows.

(2) Assuming $(B, q)$ does not reject $(a, \sigma)$ we aim for a contradiction. By assumption, there exists $q' \leq q$ and $B' \in [\text{depth}_{A}(a), A]$ such that $[a, B'] \times [q'|\sigma] \subseteq O$. Since $B' \subseteq B \subseteq A$ and $[a, B'] \not= \emptyset$, by the transitivity of $\leq$ and axiom (A3) (2) of topological Ramsey spaces, there exists $B'' \in [\text{depth}_{A}(a), A]$ such that $\emptyset \not= [a, B''] \subseteq [a, B']$. Thus $[a, B''] \times [q'|\sigma] \subseteq O$ and hence $(B'', q')$ accepts $(a, \sigma)$. This contradicts that $(A, p)$ rejects $(a, \sigma)$.

(5) If $(A, p)$ accepts $(a, \tau)$, then clearly $(A, p)$ accepts $(a, \sigma)$. So we suppose $(A, p)$ rejects $(a, \tau)$. Assuming $(A, p)$ does not reject $(a, \sigma)$, we aim for a contradiction. By assumption, there exists $q \leq_{\sigma} p$ and $B \in [\text{depth}_{A}(a), A]$ such that $[a, B] \times [q|\sigma] \subseteq \emptyset$. Note that $q|\sigma \leq_{\tau} p$: $[q|\sigma] \subseteq [p|\tau]$ implies that $\tau(i) \subseteq \sigma(i)$ or $\sigma(i) \subseteq \tau(i)$ and there is no branching between $\sigma(i)$ and $\tau(i)$, for $i \in \text{dom}(\tau)$. Thus $q \leq_{\sigma} p$ implies $q \leq_{\tau} p$ and hence $q|\sigma \leq_{\tau} p$. So we have $q|\sigma \leq_{\tau} p$ and $B \in [\text{depth}_{A}(a), A]$ such that $[a, B] \times [(q|\sigma)|\tau] \subseteq \emptyset$, contradicting that $(A, p)$ rejects $(a, \tau)$.

(6) By assumption, $[a, A] \times [p|\sigma] \not= \emptyset$. We assume $(A, p)$ does not reject $(a, \sigma)$ and aim for a contradiction. Let $B \in [\text{depth}_{A}(a), A]$, $q \leq_{\sigma} p$ be such that $(B, q)$ accepts $(a, \sigma)$. Then $\exists i \ (q|\tau_{i}) \leq_{\tau} p$. So $(B, q|\tau_{i})$ accepts $(a, \tau_{i})$, contradicting that $(A, p)$ rejects $(a, \tau_{i})$.

\[ \square \]

**Lemma 2.6.** Let $A \in \mathcal{R}$, $a \in \mathcal{AR}[\emptyset, A]$, $p \in \mathcal{P}_{\omega}$. Then there exists $q \leq_{\text{depth}_{A}(a)} p$ and $B \in [\text{depth}_{A}(a), A]$ such that $\forall m < \omega \ \forall \sigma \in I(m, q)$

$$
\forall b \in \mathcal{AR} \ (\text{depth}_{A}(a) \leq \text{depth}_{B}(b) \leq m) \Rightarrow (B, q) \text{ decides } (b, \sigma).
$$

**Proof.** We construct fusion sequences $([n_{k}, A_{k}])_{k < \omega}$ and $(p_{k})_{k < \omega}$ such that for $k \geq 0$:

(i) $A_{0} = A$, $p_{0} = p$, and $n_{k} = \text{depth}_{A}(a) + k$;
(ii) $p_{k+1} \leq^{n_k} p_k$, $A_{k+1} \in \{n_k, A_k\}$;

(iii) for every $i \leq k$ and every element $(\sigma, b)$ in the set

$$S_i = l(n_i, p_k) \times \{b \in \mathcal{AR} : \text{depth}_{A_i}(b) = n_i\}$$

$$(A_{k+1}, p_{k+1}) \text{ decides } (b, \sigma).$$

Suppose we have $A_k, p_k$. Note $S_k$ is finite by (A2) (1). Apply Lemma 2.5 (4) repeatedly for elements of $S_k$, we obtain $A_{k+1} \in \{n_k, A_k\}$ and $p_{k+1} \leq^{n_k} p_k$ such that $(A_{k+1}, p_{k+1})$ decides every $(b, \sigma) \in S_k$. Then by the induction hypothesis on $(A_k, p_k)$ and Lemma 2.5 (1), $(A_{k+1}, p_{k+1})$ satisfies (iii).

Now let $B, q$ be the fusions of the sequences. Then $q \leq^{\text{depth}_A(a)} p$ and $B \in [\text{depth}_A(a), \omega]$. We check that $B, q$ satisfy the lemma: Let $m \in \omega$, $\sigma \in l(m, q)$ and $b \in \mathcal{AR}$ with depth$_A(a) \leq \text{depth}_B(b) \leq m$. Let $n_k = \text{depth}_B(b)$. So $n_k = \text{depth}_{A_i}(b)$. We can find $\tau \in l(n_k, q)$ such that $[q|\sigma] \subseteq [q|\tau]$. By construction, $(A_{k+1}, p_{k+1})$ decides $(b, \tau)$, so it decides $(b, \sigma)$ by Lemma 2.5 (5). Then by Lemma 2.5 (3), $(B, q)$ decides $(b, \sigma)$ as required.

\[\square\]

### 2.1.1 Digression: Abstract Galvin Lemma

In order to prove Theorem 2.10, we slightly strengthen the Abstract Galvin Lemma [20, Thm. 5.15]. The proof follows easily from that in [20]. We write it down here for completeness.

**Theorem 2.7** (Abstract Galvin Lemma). Let $\mathcal{R}$ be a topological Ramsey space. For every family $\mathcal{F} \subseteq \mathcal{AR}$, every $A \in \mathcal{R}$ and every $a \in \mathcal{AR}$ with depth$_A(a) < \infty$ there exists $B \in [\text{depth}_A(a), \omega]$ such that either $\mathcal{F} \cap \mathcal{AR}[a, B] = \emptyset$ or every $X \subseteq [a, B]$ has an initial segment in $\mathcal{F}$.

**Proof.** As defined in [20, Def. 5.2], we say a subset $X \subseteq \mathcal{R}$ is Ramsey if for every basic set $[a, X] \neq \emptyset$ there exists $Y \subseteq [a, X]$ such that $[a, Y] \subseteq X$ or $[a, Y] \cap X = \emptyset$. By [20, Cor. 5.11], every metrically Borel set of $\mathcal{R}$ is Ramsey. Therefore, the open set $\mathcal{O} = \{X \subseteq \mathcal{R} : (\exists n)(r_n(X) \in \mathcal{F})\}$ is Ramsey. So, for the given $a$ and $A$, there exists $C \subseteq [a, A]$ such that either of the following holds: If $[a, C] \subseteq \mathcal{O}$, then every $X \subseteq [a, C]$ has an initial segment in $\mathcal{F}$. Otherwise $[a, C] \cap \mathcal{O} = \emptyset$, so $\mathcal{F} \cap \mathcal{AR}[a, C] = \emptyset$. Finally, by axiom (A3) (2) of topological Ramsey spaces, we can find $B \in [\text{depth}_A(a), \omega]$ such that $[a, B] \subseteq [a, C]$. Then $B$ is as required. \[\square\]

Although we only need the usual version [20, Thm. 5.17], we note that the following slightly strengthened Abstract Nash-Williams Theorem follows easily from Theorem 2.7. Recall that $\mathcal{F} \subseteq \mathcal{AR}$ is a Nash-Williams family if $s \not\subseteq t$ for every pair of distinct $s, t \in \mathcal{F}$.

**Theorem 2.8** (Abstract Nash-Williams Theorem). Let $\mathcal{R}$ be a topological Ramsey space. For every Nash-Williams family $\mathcal{F} \subseteq \mathcal{AR}$, every partition $\mathcal{F} = F_0 \sqcup F_1$, and every basic set $[a, A] \neq \emptyset$, there exists $B \in [\text{depth}_A(a), \omega]$ and $i \in 2$ such that $F_i[a, B] = \emptyset$.

The digression ends. We return to open subsets of $\mathcal{R} \times \mathbb{R}^N$ where $\mathcal{R}$ is a topological Ramsey space satisfying (L4).

To prove Theorem 2.10, we need a technical lemma.
Lemma 2.9. Suppose \((b_i)_{i<\omega} \subseteq \mathcal{AR}\) and \((B_i)_{i<\omega} \subseteq \mathcal{R}\) are sequences such that for \(i < j < \omega\), \(b_i \neq b_j\) and
\[
\text{depth}_{B_i}(b_i) \leq \text{depth}_{B_j}(b_j) \land B_{i+1} \in [\text{depth}_{B_i}(b_{i+1}), B_i].
\]
Then \((\text{depth}_{B_i}(b_{i+1}))_{i<\omega}\) is unbounded, hence \((\text{depth}_{B_i}(b_{i+1}), B_i)_{i<\omega}\) is a fusion sequence.

Proof. It is sufficient to check that \((\text{depth}_{B_i}(b_{i+1}))_{i<\omega}\) is unbounded. We assume otherwise and aim for a contradiction.

Since
\[
\text{depth}_{B_i}(b_{i+1}) = \text{depth}_{B_{i+1}}(b_{i+1}) \leq \text{depth}_{B_{i+2}}(b_{i+2}),
\]
\((\text{depth}_{B_i}(b_{i+1}))_{i<\omega}\) is increasing. So it must have a tail consisting of a constant. We assume without loss of generality that for some \(n \in \omega\),
\[
\forall i < \omega \quad \text{depth}_{B_i}(b_{i+1}) = n.
\]
Then, since \(B_{i+1} \in [\text{depth}_{B_i}(b_{i+1}), B_i]\),
\[
\forall i < \omega \quad (B_{i+1} \leq B_i) \land (r_n(B_{i+1}) = r_n(B_i)).
\]
In particular, \(\forall i < \omega\),
\[
r_n(B_i) = r_n(B_0).
\]
Hence \(\forall i < \omega\),
\[
\text{depth}_{B_i}(b_{i+1}) = n \text{ so } \text{depth}_{B_0}(b_{i+1}) = n.
\]
Then
\[
(b_i)_{i<\omega} \subseteq \{b \in \mathcal{AR} : b \leq \text{fin } r_n(B_0)\},
\]
but \(\{b \in \mathcal{AR} : b \leq \text{fin } r_n(B_0)\}\) is finite by (A2) (1). A contradiction.

Theorem 2.10. Let \(\mathcal{O} \subseteq \mathcal{R} \times \mathbb{R}^N\) be an open set. For every \(p \in \mathcal{P}_\omega\), \(A \in \mathcal{R}\) and \(a \in \mathcal{AR}[\emptyset, A]\) there exists \(q \leq p\) and \(B \in [\text{depth}_A(a), A]\) such that \([a, B] \times [q] \subseteq \mathcal{O}\) or \([a, B] \times [q] \cap \mathcal{O} = \emptyset\).

Proof. By Lemma 2.6, shrinking \(p\) and \(A\), we may assume for all \(b \in \mathcal{AR}\) and \(m \in \omega\),
\[
\forall \sigma \in l(m, p) \quad (\text{depth}_A(a) \leq \text{depth}_A(b) \leq m \Rightarrow (A, p) \text{ decides } (b, \sigma)). \quad (*)
\]
In particular,
\[
\forall \sigma \in l(\text{depth}_A(a), p) \quad (A, p) \text{ decides } (a, \sigma).
\]
If there is \(\sigma \in l(\text{depth}_A(a), p)\) such that \((A, p)\) accepts \((a, \sigma)\), then \([a, A] \times [p]\sigma] \subseteq \mathcal{O}\), and \(B = A, q = p\sigma\) satisfy the theorem. So we assume
\[
\forall \sigma \in l(\text{depth}_A(a), p) \quad (A, p) \text{ rejects } (a, \sigma).
\]
Then by Lemma 2.5 (6),
\[
(A, p) \text{ rejects } (a, \emptyset).
\]
We aim to find \(q \leq p\) and \(B \in [\text{depth}_A(a), A]\) such that \([a, B] \times [q] \cap \mathcal{O} = \emptyset\).
We achieve this by constructing barriers on \([a, A]\) of higher and higher rank, shrinking \(A\) and \(p\) correspondingly such that \((A, p)\) rejects \(c, \emptyset\) for every \(c\) in the barriers, as follows.
We construct sequences \((A_k)_{k<\omega} \subseteq \mathcal{R}\), \((p_k)_{k<\omega} \subseteq \mathcal{P}\) and \((\mathcal{F}_k)_{k<\omega}\) such that \(\forall k < \omega\),
Case 1. \(a_0 = a, A_0 = A, p_0 = p, F_0 = \{a\};\)

(2) \(n_k = \min\{\text{depth}_{A_k}(b) : b \in F_k\};\)

(3) \(A_{k+1} \in [n_k, A_k], p_{k+1} \leq n_k p_k;\)

(4) \(F_{k+1}\) is a barrier on \([a, A_{k+1}];\)

(5) \((\forall b \in F_k \cap AR[a, A_{k+1}] \exists c \in F_{k+1} b \sqsubseteq c)\) and

\((\forall c \in F_{k+1} \cap AR[a, A_{k+1}] \exists b \in F_k b \sqsubseteq c);\)

(6) \((A_k, p_k)\) rejects \((b, \emptyset)\) for \(b \in F_k \cap AR[a, A_k].\)

Suppose we have constructed \(a_k, A_k, p_k\) and \(F_k\). We construct sequences 
\((b_i)_{0 < i < \omega}, (B_i)_{i < \omega}, (q_i)_{i < \omega}\) and \((G_i)_{i < \omega}\) such that \(\forall i < \omega\)

(i) \(B_0 = A_k, q_0 = p_k;\)

(ii) \(B_{i+1} \in [\text{depth}_{B_i}(b_{i+1}), B_i], q_{i+1} \leq \text{depth}_{B_i}(b_{i+1}) q_i;\)

(iii) \(b_{i+1} \in F_k \setminus \{b_1, \ldots, b_i\}\) such that \(\text{depth}_{B_i}(b_{i+1})\) is minimal;

(iv) \(G_{i+1}\) is a barrier on \([b_{i+1}, B_{i+1}];\)

(v) \((B_{i+1}, q_{i+1})\) rejects \((c, \emptyset)\) for \(c \in G_{i+1} \cap AR[b_{i+1}, B_{i+1}];\)

(vi) \((B_{i+1}, q_{i+1})\) rejects \((b, \emptyset)\) for \(b \in F_k \cap AR[a, A_{i+1}].\)

Suppose we have \(B_i, q_i\). Let \(b_{i+1}\) be as in (iii). By the induction hypothesis (6), \((A_k, p_k)\) rejects \((b_{i+1}, \emptyset)\). So by Lemma 2.5 (2),

\((B_i, q_i)\) rejects \((b_{i+1}, \emptyset)\).

For \(c \in AR\), let

\[O_c^{i+1} = \bigcup \{[q_i]|\tau] : [c, B_i] \times [q_i]|\tau] \subseteq O\} \]

\[B_{i+1} = \{c \in AR|b_{i+1}, B_i] : (b_{i+1} \neq c) \land (O_c^{i+1} \neq \emptyset)\} \]

Applying Theorem 2.7 to \(B_{i+1}\) and \(b_{i+1}, B_i\), we have two cases.

Case 1. \(\exists B_{i+1} \in [\text{depth}_{B_i}(b_{i+1}), B_i]\) such that \(B_{i+1} \cap AR[b_{i+1}, B_{i+1}] = \emptyset.\) Then 
\([b_{i+1}, B_{i+1}] \times [q_i]|\bigcap O = \emptyset.\) Otherwise \(\exists(X, e) \in [b_{i+1}, B_{i+1}] \times [q_i]|\bigcap O.\) Since \(O\) is open, there is \(e \subseteq X\) and \(\sigma \subseteq^* e\) such that \(b_{i+1} \subseteq c\) and \([c] \times [q_i]|\sigma \subseteq O.\) So \(O_c^{i+1} \neq \emptyset\) and hence \(c \in B_{i+1} \cap \text{AR}[b_{i+1}, B_{i+1}],\) contradicting \(B_{i+1} \cap \text{AR}[b_{i+1}, B_{i+1}] = \emptyset.\)

Let \(q_{i+1} = q_i\) and \(G_{i+1}\) be an arbitrary barrier on \([b_{i+1}, B_{i+1}]\) but \(G_{i+1} \neq \{b_{i+1}\}.\) In particular, \((B_{i+1}, q_{i+1})\) rejects \((c, \emptyset)\) for all \(c \in \text{AR}[b_{i+1}, B_{i+1}].\)

Case 2. \(\exists B_{i+1} \in [\text{depth}_{B_i}(b_{i+1}), B_i]\) such that there is a barrier \(G_{i+1} \subseteq B_{i+1} \cap \text{AR}[b_{i+1}, B_{i+1}]\) on \([b_{i+1}, B_{i+1}].\) By (L4) and Lemma 2.2, we further assume that there exist \(q_{i+1} \leq \text{depth}_{B_i}(b_{i+1}) q_i\) and \(G \subseteq [q_{i+1}]\) clopen such that

\([q_{i+1}] \cap O_c^{i+1} = G\) for all \(c \in G_{i+1} \cap AR[b_{i+1}, B_{i+1}].\)
Claim 2.10.1. $G = \emptyset$.

Proof of Claim 2.10.1. Otherwise by Lemma 2.3, there is $q' \leq q_{i+1}$ such that $[q'] \subseteq G \subseteq O^+_{c+1}$. Then

$$[b_{i+1}, B_{i+1}] \times [q'] \subseteq \bigcup \{[c, B_{i+1}] \times [q] : c \in G_{i+1} \cap AR[b_{i+1}, B_{i+1}]\} \subseteq \bigcup \{[c, B_i] \times [q] : c \in G_{i+1} \cap AR[b_{i+1}, B_{i+1}]\} \subseteq \mathcal{O}.$$ 

This contradicts that $(B_i, q_i)$ rejects $(b_{i+1}, \emptyset)$. \hfill \Box

Claim 2.10.2. $(B_{i+1}, q_{i+1})$ rejects $(c, \emptyset)$ for $c \in G_{i+1} \cap AR[b_{i+1}, B_{i+1}]$.

Proof of Claim 2.10.2. Let $c \in G_{i+1} \cap AR[b_{i+1}, B_{i+1}]$. Since $q_{i+1} \leq p$, by ($\ast$) and Lemma 2.5 (5), we can find $m$ large enough such that

$$\forall \sigma \in l(m, q_{i+1}) \quad (A, p) \text{ decides } (c, \sigma).$$

Now let $\sigma \in l(m, q_{i+1})$. By Lemma 2.5 (1) and (2),

$$(B_{i+1}, q_{i+1}) \text{ decides } (c, \sigma) \text{ in the same way as } (A, p).$$

If $(A, p)$ accepts $(c, \sigma)$, then $[c, A] \times [p|\sigma] \subseteq \mathcal{O}$ hence $[q_{i+1}|\sigma] \subseteq O^+_{c+1}$. This contradicts that $[q_{i+1}] \cap O^+_{c+1} = \emptyset$. Therefore, $(A, p)$ and $(B_{i+1}, q_{i+1})$ must reject $(c, \sigma)$ for all $\sigma \in l(m, q_{i+1})$. Then by Lemma 2.5 (6), $(B_{i+1}, q_{i+1})$ rejects $(c, \emptyset)$. \hfill \Box

This finishes the construction of $(b_i), (B_i), (q_i)$ and $(G_i)$. By construction (i)-(iv) hold. In both Case 1 and Case 2, (v) holds. (vi) holds by the induction hypothesis (6), Lemma 2.5 (2) and the fact that $B_{i+1} \leq A_k$ and $q_{i+1} \leq p_k$.

By (ii), (iii) and Lemma 2.9, $(\text{depth}_{B_i}(b_{i+1}))_{i<\omega} \subseteq \omega$ is increasing and unbounded, so $(\text{depth}_{B_i}(b_{i+1}), B_i)_{i<\omega}$ and $(q_i)_{i<\omega}$ are fusion sequences. Let $A_{k+1}, p_{k+1}$ be the fusions of the sequences. Let

$$\mathcal{F}_{k+1} = \bigcup \{G_i : b_i \in \mathcal{F}_k \cap AR[a, A_{k+1}]\}.$$ 

This finishes the construction of $(A_k), (p_k)$ and $(\mathcal{F}_k)$. Let us check that (1)-(6) hold.(1) and (2) hold by construction

(3) holds by (ii) since $B_0 = A_k$ and $\text{depth}_{B_0}(b_1) = n_k$ by (2) and (iii).

(4) $\mathcal{F}_k$ is a barrier on $[a, A_k]$ by the induction hypothesis (4). Moreover, each $G_{i+1}$ is a barrier on $[b_i, A_{k+1}]$ by (iv). So $\mathcal{F}_{k+1}$ is a barrier on $[a, A_{k+1}]$.

(5) holds since each $G_{i+1} \neq \{b_{i+1}\}$.

(6) follows from (vi) and Lemma 2.5 (2).

By (2) and (3), $(n_k)_{k<\omega}$ is increasing in $k$. Again Lemma 2.9 gives that $(n_k)_{k<\omega}$ is unbounded, so $(\{n_k, A_k\})_{k<\omega}$ and $(p_k)_{k<\omega}$ are fusion sequences. Let $B, q$ be the fusions of the sequences. Then by (6) and Lemma 2.5 (2),

$$\forall k < \omega \forall c \in \mathcal{F}_k \cap AR[a, B] \quad (B, q) \text{ rejects } (c, \emptyset).$$
Claim 2.10.3. \([a, B] \times [q] \cap \mathcal{O} = \emptyset\).

Proof of Claim 2.10.3. Suppose otherwise and aim for a contradiction. If \((X, \varepsilon) \in [a, B] \times [q] \cap \mathcal{O}\), then there exist \(b \subseteq X\) and \(\sigma \subseteq^* \varepsilon\) such that
\[
a \subseteq b \land [b, B] \times [q] \sigma \subseteq \mathcal{O}.
\]
By (5), we may find \(k\) large enough such that
\[
\exists c \in F_k \cap \mathcal{AR}[a, B] \quad b \subseteq c.
\]
So \([c, B] \times [q] \sigma \subseteq \mathcal{O}\) contradicting that \((B, q)\) rejects \((c, \emptyset)\).

2.2 Souslin-measurable Subsets of \(\mathcal{R} \times \mathbb{R}^N\)

In this subsection, we extend the result in the previous subsection from open subsets to all Souslin-measurable subsets, by adapting the results in [20, §9] to abstract topological Ramsey spaces parametrized by infinite sequences of perfect trees.

Definition 2.11. A subset \(\mathcal{X} \subseteq \mathcal{R} \times \mathbb{R}^N\) is perfectly Ramsey if for every \(p \in \mathcal{P}_\omega\), \(A \in \mathcal{R}\), and \(a \in \mathcal{AR}[\emptyset, A]\) there exists \(B \in [\text{depth}_A(a), A]\) and \(q \leq p\) such that \([a, B] \times [q] \subseteq \mathcal{X}\) or \([a, B] \times [q] \cap \mathcal{X} = \emptyset\).

So we can rephrase Theorem 2.10 as follows.

Theorem 2.12. Every open subset of \(\mathcal{R} \times \mathbb{R}^N\) is perfectly Ramsey.

Lemma 2.13. The perfectly Ramsey subsets of \(\mathcal{R} \times \mathbb{R}^N\) form a \(\sigma\)-field.

Proof. It is straightforward to check that the collection of perfectly Ramsey subsets form a field. We check that it is closed under countable union. Suppose \((\mathcal{X}_m)_{m<\omega}\) is a sequence of perfectly Ramsey sets. Let \(\mathcal{X} = \bigcup_{m<\omega} \mathcal{X}_m\). Without loss of generality, \(\mathcal{X}_m \subseteq \mathcal{X}_{m+1}\). Let \(p \in \mathcal{P}_\omega\), \(A \in \mathcal{R}\), and \(a \in \mathcal{AR}[\emptyset, A]\). We aim to find \(B \in [\text{depth}_A(a), A]\) and \(q \leq p\) such that \([a, B] \times [q]\) is included in or disjoint from \(\mathcal{X}\).

We construct fusion sequences \([(n_k, A_k)]_{k<\omega}\) and \((p_k)_{k<\omega}\) such that for \(k \geq 0\):

(i) \(A_0 = A, p_0 = p,\) and \(n_k = \text{depth}_A(a) + k\);

(ii) \(p_{k+1} \leq^k p_k, A_{k+1} \in [n_k, A_k]\);

(iii) for every element \((\sigma, b)\) in the set
\[
S_k = l(k, p_k) \times \{b \in \mathcal{AR} : \text{depth}_{A_k}(b) = n_k\}
\]

either \([b, A_{k+1}] \times [p_{k+1}] \sigma \subseteq \mathcal{X}_k\) or \([b, A_{k+1}] \times [p_{k+1}] \sigma \cap \mathcal{X}_k = \emptyset\).

Suppose we have \(p_k, A_k\). Since \(\mathcal{X}_k\) is perfectly Ramsey, and the set \(S_k\) is finite by (A2)(1), we can shrink \(p_k, A_k\) finitely many times to obtain \(p_{k+1}, A_{k+1}\).

Let \(A_\infty = \lim A_m\) and \(p_\infty = \bigcap_{m<\omega} p_m\). Thus \(A_\infty \in [\text{depth}_A(a), A]\) and \(p_\infty \leq p\). Moreover, for every \(m < \omega, \sigma \in l(m, p_\infty)\) and \(b\) with \(\text{depth}_{A_\infty}(b) = \text{depth}_A(a) + m\),
\[
[b, A_\infty] \times [p_\infty] \sigma \subseteq [b, A_{m+1}] \times [p_{m+1}] \sigma.
\]
which is included in or disjoint from $X_m$.

By construction, the set $X \cap [a, A_\infty] \times [p_\infty]$ is open in $[a, A_\infty] \times [p_\infty]$ with respect to the subspace topology. Therefore, there exists an open subset $O \subseteq R \times R^N$ such that $X \cap [a, A_\infty] \times [p_\infty] = O \cap [a, A_\infty] \times [p_\infty]$. By Theorem 2.12, $O$ is perfectly Ramsey. Hence we can find $B \in [\text{depth}_A(a), A_\infty]$ and $q \leq p_\infty$ such that $[a, B] \times [q] \subseteq O$ or $[a, B] \times [q] \cap O = \emptyset$. On the other hand, $[a, B] \times [q] \subseteq [a, A_\infty] \times [p_\infty]$, so we must have $[a, B] \times [q] \subseteq X$ or $[a, B] \times [q] \cap X = \emptyset$.

**Theorem 2.14.** The field of perfectly Ramsey subsets of $R \times R^N$ is closed under the Souslin operation.

**Proof.** Let $X_v (v \in [\omega]^{<\omega})$ be a given Souslin scheme of perfectly Ramsey subsets of $R \times R^N$. Without loss of generality, we identify $[\omega]^{<\omega}$ with the set of finite strictly increasing sequences, $[\omega]^{<\omega}$ with the set of infinite strictly increasing sequences, and assume $X_u \subseteq X_v$ whenever $v \subseteq u$. Let $p \in P_\omega$, $A \in R$, and $a \in AR[\emptyset, A]$. Let $X = \bigcup_{f \in P_\omega} \bigcap_{n \in \omega} X_{fn}$. We aim to find $B \in [\text{depth}_A(a), A]$ and $q \leq p$ such that $[a, B] \times [q] \subseteq X$ or $[a, B] \times [q] \cap X = \emptyset$, thus showing that $X$ is also perfectly Ramsey.

For $v \in [\omega]^{<\omega}$, let

$$X^*_v = \bigcup_{f \supseteq v, n \in \omega} X_{fn}.$$ 

So $X^*_v \subseteq X_v$ and $X^*_v \subseteq X^*_u$ whenever $v \subseteq u$. We build fusion sequences $([n_k, A_k])_{k < \omega}$ and $(p_k)_{k < \omega}$ such that for $k \geq 0$:

1. $A_0 = A$, $p_0 = p$, and $n_k = \text{depth}_A(a) + k$;
2. $p_{k+1} \leq p_k$, $A_{k+1} \in [n_k, A_k]$;
3. for every element $(\sigma, v, b)$ in the set

   $$S_k = \{(k, p_k) \times [k]^{<\omega} \times \{b \supseteq a : \text{depth}_A(b) = n_k\},$$

   either

   (1) $[b, A_{k+1}] \times [p_{k+1}[\sigma] \cap X^*_v = \emptyset$; or
   (2) there does not exist $q \leq \sigma$ $p_{k+1}$ and $B \in [n_k, A_{k+1}]$ with $[b, B] \times [q[\sigma] \cap X^*_v = \emptyset$.

Suppose we have constructed $A_k, p_k$. Let $S_k$ be enumerated as $(\sigma_i, v_i, b_i)_{i < L}$ where $L < \omega$ by (A2) (1). We can construct sequences $([n_k, B_i])_{i \leq L}$ and $(q_i)_{i \leq L}$ such that for $l < L$:

(i) $B_0 = A_k$ and $q_0 = p_k$;
(ii) $q_{i+1} \leq q_i$, $B_{i+1} \in [n_k, B_i]$;
(iii) either $[b_i, B_{i+1}] \times [q_{i+1}[\sigma] \cap X^*_v = \emptyset$, or $B_{i+1} = B_i$, $q_{i+1} = q_i$ and there does not exist $q \leq q_i[\sigma]$ and $B \in [n_k, B_i]$ such that $[b_{i+1}, B] \times [q] \cap X^*_v = \emptyset$.

Let $A_{k+1} = B_L$ and $p_{k+1} = q_L$. This finishes the construction of $([n_k, A_k])_{k < \omega}$ and $(p_k)_{k < \omega}$. Then let $A_\infty$ and $p_\infty$ be the fusions.
For \( v \in [\omega]^{<\omega} \), let
\[
\Psi(X^*_v) = \bigcup [b, A_\infty) \times [p_\infty] \cdot \sigma \colon [b, A_\infty) \times [p_\infty] \cap X^*_v = \emptyset \land a \subseteq b, \]
\[
\Phi(X^*_v) = (X_v \cap [a, A_\infty) \times [p_\infty]) \setminus \Psi(X^*_v), \text{ and}
\]
\[
M_v = \Phi(X^*_v) \setminus \bigcup_{l > \max v} \Phi(X^*_{v-l}).
\]
In particular, \( M_v \cap \Psi(X^*_v) = \emptyset \).

\[
X^*_v = \bigcup_{l > \max v} X^*_{v-l} \subseteq \bigcup_{l > \max v} X_{v-l}, \text{ and}
\]
\[
X^*_v \cap [a, A_\infty) \times [p_\infty] \subseteq \bigcup_{l > \max v} (X^*_{v-l} \cap [a, A_\infty) \times [p_\infty])
\]
\[
\subseteq \bigcup_{l > \max v} \Phi(X^*_{v-l}).
\]

So
\[
M_v \subseteq \Phi(X^*_v) \setminus \Phi(X^*_{v} \cap [a, A_\infty) \times [p_\infty]) = \Phi(X^*_v) \setminus X^*_v.
\]

Note that \( \Psi(X^*_v) \) is open in \([a, A_\infty) \times [p_\infty]\) with respect to the subspace topology, so as in the proof of Lemma 2.13, we can find an open (and hence perfectly Ramsey, by Theorem 2.12) set \( O \subseteq R \times R \) such that \( \Psi(X^*_v) = O \cap ([a, A_\infty) \times [p_\infty]) \).

Let \( O^c = R \times R \setminus O \). Therefore \( \Phi(X^*_v) = (X_v \cap O^c) \cap [a, A_\infty) \times [p_\infty] \), where \( X_v \cap O^c \) is perfectly Ramsey.

We say a set \( Y = [a, A_\infty) \times [p_\infty] \) is perfectly Ramsey inside \([a, A_\infty) \times [p_\infty]\) if it is the intersection of a perfectly Ramsey set with \([a, A_\infty) \times [p_\infty]\). The definition of perfectly Ramsey null inside is similar. So by the arguments above, \( \Psi(X^*_v) \) and \( \Phi(X^*_v) \) are both perfectly Ramsey inside \([a, A_\infty) \times [p_\infty]\). Since the perfectly Ramsey sets form a \( \sigma \)-field (Lemma 2.13), \( M_v \) is also perfectly Ramsey inside \([a, A_\infty) \times [p_\infty]\).

**Claim 2.14.1.** For every \( v \in [\omega]^{<\omega} \), \( M_v \) is perfectly Ramsey null inside \([a, A_\infty) \times [p_\infty]\).

**Proof.** Suppose \( \emptyset \neq [b, B] \times [q] \subseteq [a, A_\infty) \times [p_\infty] \cap M_v \) for some \( v \in [\omega]^{<\omega} \), \( b \in AR \) and \( B \in R \). We aim for a contradiction.

Pick \( Y \in [b, B] \), so \( b \subseteq Y \subseteq B \subseteq A_\infty \). By (A2) (1), we may pick \( l \geq |b| \) large enough such that, for \( b' = r_l(Y) \),

\[
\text{depth}_{A_\infty}(b') = n_k,
\]
where \( k \) is in turn large enough such that \( v \in [k]^{<\omega} \). Thus \( [b', Y'] \neq \emptyset \) and \( Y' \subseteq A_\infty \), so by (A3) (2) there exists \( Y' \in [n_k, A_\infty) \) such that \( [b', Y'] \subseteq [b', Y] \).

So
\[
[b', Y'] \times [q] \subseteq [b', Y] \times [q] \subseteq [b, B] \times [q] \subseteq M_v,
\]
which is disjoint from \( X^*_v \). Pick \( \sigma \in l(k, p_\infty) \) such that \( \sigma(i) \in q(i) \) for all \( i < k \), so \( q \leq \sigma \) \( p_\infty \).

Thus, for \( \sigma \in l(k, p_\infty) \), \( v \in [k]^{<\omega} \) and \( b' \upharpoonright a \) with \( \text{depth}_{A_\infty}(b') = n_k \), we have \( q \leq \sigma p_\infty \) and \( Y' \in [n_k, A_\infty) \) such that \( [b', Y'] \times [q] \cap X^*_v = \emptyset. \) By the construction of \( A_\infty \) and \( p_\infty \), (1) must hold. So \([b', A_\infty) \times [p_\infty] \cap X^*_v = \emptyset \).

Thus
\[
[b', Y'] \times [q] \subseteq M_v \cap \Psi(X^*_v),
\]
contradicting \( M_v \cap \Psi(X^*_v) = \emptyset \). \( \square \)
Therefore we can build fusion sequences \( ([n_k, B_k])_{k < \omega} \) and \( (q_k)_{k < \omega} \) such that for \( k \geq 0 \),

(i) \( B_0 = A_\infty, q_0 = p_\infty \) and \( n_k = \text{depth}_A(a) + k \);

(ii) \( q_{k+1} \leq^{\leq} q_k, B_{k+1} \in [n_k, B_k] \);

(iii) \( \forall \tau \in l(k, q_k) \forall v \in [k]^{<\omega} \forall b \ni a \) with \( \text{depth}_{B_k}(b) = n_k \); \( [b, B_{k+1}] \times [q_{k+1}] \cap \mathcal{M}_v = \emptyset \).

Let \( B_\infty \) and \( q_\infty \) be the fusions of the sequences. We check that \( [a, B_\infty] \times [q_\infty] \cap \mathcal{M}_v = \emptyset \) for all \( v \in [\omega]^{<\omega} \). Suppose \( (X, \varepsilon) \in [a, B_\infty] \times [q_\infty] \) and \( v \in [\omega]^{<\omega} \). We can find \( k > \max v \) such that there exist \( b \subseteq X \) and \( \sigma \subseteq^* \varepsilon \) with

\[
\text{depth}_{B_\infty}(b) = n_k \land \sigma \in l(k, q_\infty).
\]

As \( [b, B_{k+1}] \times [q_{k+1}] \sigma \cap \mathcal{M}_v = \emptyset \) by construction, we have \( (X, \varepsilon) \notin \mathcal{M}_v \).

**Claim 2.14.2.** \( [a, B_\infty] \times [q_\infty] \cap \mathcal{X}_0^* = [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}_0^*) \).

*Proof of Claim 2.14.2.* Since \( \mathcal{X}_0^* \subseteq \Phi(\mathcal{X}_0^*) \), it is clear that \( [a, B_\infty] \times [q_\infty] \cap \mathcal{X}_0^* \subseteq [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}_0^*) \). To prove the other inclusion, we pick \( \bar{x} \in [a, B_\infty] \times [q_\infty] \cap \Phi(\mathcal{X}_0^*) \). Recall that \( [a, B_\infty] \times [q_\infty] \cap \mathcal{M}_0 = \emptyset \) and \( \mathcal{M}_0 = \Phi(\mathcal{X}_0^0) \setminus \bigcup_{k < \omega} \Phi(\mathcal{X}_0^k) \). So there exists \( l_0 \in \omega \) such that \( \bar{x} \in \Phi(\mathcal{X}_{l_0}^0) \). By repeating this process, we find \( v = \{ l_0 < l_1 < \cdots \} \in [\omega]^{<\omega} \) such that \( \bar{x} \in \Phi(\mathcal{X}_{l_k}^0) \) for all \( k < \omega \). Hence \( \bar{x} \in \bigcap_{m < \omega} \mathcal{X}_{l_k}^0 \subseteq X = \mathcal{X}_0^0 \).

As \( \Phi(\mathcal{X}_0^0) \) is perfectly Ramsey inside \( [a, A_\infty] \times [p_\infty], B_\infty \in [\text{depth}_A(a), A_\infty] \) and \( q_\infty \leq p_\infty \), there exists \( B \in [\text{depth}_A(a), B_\infty] \subseteq [\text{depth}_A(a), A] \) and \( q \leq q_\infty \leq p \) such that \( [a, B] \times [q] \subseteq \Phi(\mathcal{X}_0^0) \) or \( [a, B] \times [q] \cap \Phi(\mathcal{X}_0^0) = \emptyset \). As \( X = \mathcal{X}_0^0 \), by Claim 2.14.2, we have \( [a, B] \times [q] \subseteq X \) or \( [a, B] \times [q] \cap X = \emptyset \), as required.

This finishes the proof of Theorem 0.2.

**Theorem 0.2** (Moderately-Abstract Parametrized Ellentuck Theorem). Let \( \mathcal{R} \) be a topological Ramsey space satisfying (L4). For every finite Souslin-measurable colouring of \( \mathcal{R} \times \mathbb{R}^N \) and for every \( A \in \mathcal{R}, a \in \mathcal{AR}[\emptyset, A] \) and \( p \in \mathcal{P}_\omega \) there exists \( B \in [a, A] \) and \( q \leq p \) such that \( [a, B] \times [q] \) is monochromatic.

In fact, we have the following.

**Corollary 2.15.** Suppose \( \mathcal{R} \) is a topological Ramsey space satisfying (L4). For every finite Souslin-measurable colouring of \( \mathcal{R} \times \mathbb{R}^N \), for every \( p \in \mathcal{P}_\omega, A \in \mathcal{R}, \) and \( a \in \mathcal{AR}[\emptyset, A] \) there exists \( B \in [\text{depth}_A(a), A] \) and \( q \leq p \) such that \( [a, B] \times [q] \) is monochromatic.

### 3 Parametrized High-dimensional Ellentuck Theorem

In order to prove the Parametrized High-dimensional Ellentuck Theorem 0.3, we show that the High-dimensional Ellentuck spaces satisfy (L4). We use methods developed in [20, 99] and the Halpern-Läuchli theorem.

From now on in this section, we fix an arbitrary integer \( k \geq 2 \).
Definition 3.1. [10]. Let $l < k$. A subset $U \subseteq \omega^l$ is isomorphic to a member of $\mathcal{E}_{k-l}$ if its structure is the same as $\omega^{\mathcal{E}(k-l)}$. More precisely, $U$ is isomorphic to a member of $\mathcal{E}_{k-1}$ if there is a $\prec$-and-$\subseteq$-preserving bijection

$$\theta: \{\vec{t} \in \vec{U} : \text{stem}(\vec{U}) \subseteq \vec{t}\} \to \omega^{\mathcal{E}(k-l)},$$

where $\vec{U}$ is the $\subseteq$-downward closure of $U$ and

$$\text{stem}(\vec{U}) = \bigcup \{\vec{s} \in \omega^{\mathcal{E}(l)} : \forall \vec{u} \in U \vec{s} \subseteq \vec{u}\}.$$

Recall from Section 1 that

$$\mathcal{AE}_k[a, A] = \{b \in \mathcal{AE}_k : (\exists n < \omega)(\exists X \in [a, a]) r_n(X) = b\},$$

and for $n < \omega$,

$$\mathcal{AE}_n^k[a, A] = \{r_n(B) : (a \subseteq B) \land (B \leq A)\}.$$

Fact 3.2. [10, Fact 19]. Let $l < k$, $n \in N^k$, $X \in \mathcal{E}_k$ and $a \in r_n[\emptyset, X]$ be given.

(1) Suppose $l \geq 1$. Let $V \subseteq r_{n+1}[a, X]$ be such that $U := \{b \setminus a : b \in V\}$ is isomorphic to a member of $\mathcal{E}_{k-l}$. Then there is a $Y \in [a, X]$ such that $r_{n+1}[a, Y] \subseteq V$.

(2) Suppose $l = 0$. Let $\omega^l = (\vec{t}_j)_{j < \omega}$ and $X = (\vec{v}_j)_{j < \omega}$ be $\prec$-increasing enumerations. Let $I \subseteq \{m \geq n : m \in N^k\}$ be an infinite subset such that

(a) for every distinct elements $m, m' \in I$, $\pi_1(\vec{t}_m) \neq \pi_1(\vec{t}_{m'})$, and

(b) for $m \in I$, there is $U_m \subseteq \{\vec{v}_j \in X : \pi_1(\vec{t}_j) = \pi_1(\vec{t}_m)\}$ such that $U_m$ is isomorphic to a member of $\mathcal{E}_{k-l}$.

Then there is a $Y \in [a, X]$ such that $\mathcal{AE}_n^k[a, Y] \subseteq \bigcup_{m \in I} U_m$.

The following fact easily follows.

Fact 3.3. If $S \subseteq \omega^l$ and there exists infinitely many $n \in \pi_1(S)$ such that $\{M \in \{m \in \omega : n < m\}^{k-1} : \{n\} \cup M \in S\}$ is isomorphic to an element of $\mathcal{E}_{k-1}$ then there is $X \in \mathcal{E}_k$ such that $X \subseteq S$.

We will be using the infinite Halpern-Läuchli theorem. Recall from Section 1, for $p \in \mathcal{P}_\omega$, $p(i)$ is the $i$th perfect tree in $p$, and $(p(i))_n$ is the $n$th level of the perfect tree $p(i)$.

Theorem 3.4. (HL$\omega$). [13]. For every $p \in \mathcal{P}_\omega$, $B \in [\omega]^\omega$ and every partition $\bigcup_{n \in B} \prod_{i < \omega} (p(i))_n = G_0 \sqcup G_1$, there exists $A \in [B]^\omega$, $q \leq p$ and $j \in 2$ such that $\bigcup_{n \in A} \prod_{i < \omega} (q(i))_n \subseteq G_j$.

Lemma 3.5. [21, Lem. 2.7]. Let $M \in \mathcal{E}_1$ and $O_l (l \in M)$ be a family of open subsets of $(2^\omega)^\omega$. For $p \in \mathcal{P}_\omega$ there exists $q \leq p$, $N \in [M]^\omega$ such that $O_l \cap [q]$ is constant for every $l \in N$.

Recall from Section 1, for $X \in \mathcal{E}_k$, $[X] = \{\max(\vec{v}) : \vec{v} \in X\} \subseteq \mathbb{N}$. Similarly, for every $S \subseteq \omega^k$, $[S] = \{\max(\zeta) : \zeta \in S\} \subseteq \mathbb{N}$.
Lemma 3.6. Suppose $A \in \mathcal{E}_k$ and $O_l (l \in [A])$ is a family of open subsets of $(2^\omega)^\omega$. Then for every $p \in \mathcal{P}_\omega$ there exists $q \leq p$ and $B \subseteq A$ such that $O_l \cap [q]$ is constant for every $l \in [B]$.

Proof. We prove by induction on $k$. When $k = 1$, the result is given by Lemma ???. So we assume $k \geq 2$. Let $A, O_l, p$ be given as in the lemma. For $m \in \omega$, let

$$S_m = \{ \bar{v} \in A : (m) \subseteq \bar{v} \} \text{ and } M = \{ m \in \omega : S_m \neq \emptyset \}.$$ 

Then for every $m \in M$, $S_m$ is isomorphic to an element of $\mathcal{E}_{k-1}$.

We construct a family of open subsets $(p_n)_{n<\omega}$ and sequences $(G_m)_{m \in M}$ such that for $m < \omega$:

1. $p_{-1} = p$, $p_m \leq^m p_m - 1$;

and for $m \in M$:

2. $G_m \subseteq [p_m]$ is open;

3. $B_m \subseteq S_m$ is isomorphic to an element of $\mathcal{E}_{k-1}$; and

4. $\forall l \in [B_m]$ $O_l \cap [p_m] = G_m$.

If $m \notin M$, let $p_m = p_m - 1$; otherwise, $p_m, G_m$, and $B_m$ exist by the induction hypothesis. This finishes the construction of $(p_m)_{m<\omega}$ and $(G_m)_{m \in M}$.

Let $p_\omega = \bigcap_{m<\omega} p_m$. Then for every $m \in M$ and $l \in [B_m]$, $O_l \cap [p_\omega] = G'_m$ where $G'_m = G_m \cap [p_\omega]$ which is open in $[p_\omega]$.

Now we have a family $G'_m (m \in M)$ of open subsets of $[p_\omega]$. By Lemma 3.5 again, there exists $q \leq p_\omega, N \subseteq [M]^\omega$ and $G' \subseteq [q]$ such that $G'_m \cap [q] = G$ for all $m \in N$. By Fact 3.3, we can find $B \in \mathcal{E}_k$ such that $B \subseteq \bigcup_{m \in N} B_m$. Then every $l \in [B]$ is in $[B_m]$ for some $m$, so $O_l \cap [q] = G'_m \cap [q] = G$, as required. $\square$

Lemma 3.7. Let $p \in \mathcal{P}_\omega, A \in \mathcal{E}_k$ and $a \in \mathcal{A}\mathcal{E}^k_{[a]_1} \subseteq [A]$ be a family of open subsets of $[p]$. Then there exists $B \in [\text{depth}_A(a), A]$ and $q \leq p$ such that $O_b \cap [q]$ is constant on $b \in \mathcal{A}\mathcal{E}^k_{[a]_1} \subseteq [a, B]$.

Proof. For $l \in [0, k)$. Let $U = \{ b \setminus a : b \in \mathcal{A}\mathcal{E}^k_{[a]_1} \subseteq [a, A] \}$. Then $U$ is isomorphic to an element of $\mathcal{E}_{k-l}$. Relabel the sets $O_b (b \in \mathcal{A}\mathcal{E}^k_{[a]_1} \subseteq [a, A])$ as $P_n (n \in [U])$ where $P_{[n]} = O_b$. Then by Lemma 3.6, there exists $V \subseteq U$ isomorphic to an element of $\mathcal{E}_{k-l}, q \leq p$ and $G \subseteq [q]$ such that $P_n \cap [q] = G$ for all $n \in [V]$. So $O_b \cap [q] = G$ for all $b \in \mathcal{A}\mathcal{E}^k_{[a]_1} \subseteq [a, A]$ with $b \setminus a \in V$. By Fact 3.2 (1) and (A3) (2), we can find $B \in [\text{depth}_A(a), A]$ such that $\mathcal{A}\mathcal{E}^k_{[a]_1} \subseteq [a, B]$.

If $l = 0$, the result similarly follows by Lemma 3.6 and Fact 3.2 (2). $\square$

Thus, we have proved that the High-dimensional Ellentuck spaces $\mathcal{E}_k (k \geq 2)$ satisfy (L4). Therefore, by Theorem 0.2 we obtain the Parametrized High-dimensional Ellentuck Theorem 0.3 in the Introduction Section 0.

Theorem 0.3 (Parametrized High-dimensional Ellentuck Theorem). For every finite Souslin-measurable colouring of $\mathcal{E}_k \times \mathbb{R}^N$ and for every $A \in \mathcal{E}_k, a \in \mathcal{A}\mathcal{R}[0, 1]$ and $p \in \mathcal{P}_\omega$ there exists $B \subseteq [a, A]$ and $q \leq p$ such that $[a, B] \times [q]$ is monochromatic.
4 The Generic Ultrafilter $B_k$

In this section we discuss properties of $B_k$, which is a generic filter for $(E_k, \subseteq \text{FIN}^\otimes k)$. The sets $\text{FIN}^\otimes k \subseteq \mathcal{P}([\omega]^k)$ are defined as follows.

**Definition 4.1.** Let $\mathcal{F}$ be the set of cofinite sets of $\omega$, i.e.
$$\mathcal{F} = \{X \subseteq \omega : \omega \setminus X \text{ is finite}\}.$$ For a sentence $\varphi(n)$, we use $\mathcal{F}$ as a quantifier:
$$(\mathcal{F}i)\varphi(i) \text{ if and only if } \exists A \in \mathcal{F} \forall i \in A \varphi(i),$$
or equivalently,
$$(\mathcal{F}i)\varphi(i) \text{ if and only if for all but finitely many } i \in \omega, \varphi(i) \text{ hold.}$$

Then let
$$\text{FIN}^\otimes k = \{F \subseteq [\omega]^k : (\mathcal{F}i_1)\cdots(\mathcal{F}i_k) \{i_1, \ldots, i_k\} \notin F\}.$$ So $\text{FIN}^\otimes 1$ can be identified with the set $\text{FIN}$ of all nonempty finite subsets of $\omega$.

For $X, Y \subseteq [\omega]^k$, let
$$X \subseteq \text{FIN}^\otimes k Y \text{ if and only if } X \setminus Y \in \text{FIN}^\otimes k.$$ Let us first recall some properties of ultrafilters in topological Ramsey spaces. We write
$$\mathcal{F}|X = \{Y \in \mathcal{F} : Y \subseteq X\}.$$ **Definition 4.2.** [7, Def. 6.2]. A subset $V \subseteq \mathcal{R}$ is an ultrafilter if the following holds.

(a) $V$ is a filter on $(\mathcal{R}, \subseteq)$, i.e.
(i) $\forall A, B \in \mathcal{R} ((A \in V) \land (A \subseteq B) \Rightarrow (B \in V));$
(ii) For every $A, B \in V$ and $a \in \mathcal{A}\mathcal{R},$
\[\left(((a, A) \neq \emptyset) \land ([a, B] \neq \emptyset) \Rightarrow (\exists C \in V) (C \in [a, A] \cap [a, B]))\right].

(b) $V$ is a maximal filter on $(\mathcal{R}, \subseteq)$: If $V'$ is a filter on $(\mathcal{R}, \subseteq)$ and $V \subseteq V'$ then $V = V'$.

(c) For every $A \in V$ and $a \in \mathcal{A}\mathcal{R}[0, A],$
(i) if $B \in [\text{depth}_A(a), A] \cap \mathcal{U}$ then $[a, B] \neq \emptyset,$
(ii) if $B \in V$, $B \subseteq A$, and $[a, B] \neq \emptyset$, then there exists $A' \in [\text{depth}_A(a), A] \cap \mathcal{V}$ such that $\emptyset \neq [a, A'] \subseteq [a, B].$

**Definition 4.3.** Let $\mathcal{R}$ be a topological Ramsey space and $\mathcal{U}$ an ultrafilter in $\mathcal{R}$. We say

- $[21]$ $\mathcal{U}$ is Nash-Williams if for every Nash-Williams family $\mathcal{G} \subseteq \mathcal{A}\mathcal{R}$ and every partition $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$ there exists $X \in \mathcal{U}$ and $i \in 2$ such that $\mathcal{G}_i \setminus X = \emptyset;$$22
• [14] $U$ is Ramsey if for all $A \in U$, $a \in \mathcal{AR}[\emptyset, A]$ and $n \in \omega$, and for every $f : \mathcal{AR}_{|a|+n} \to 2$ there exists $B \in [\text{depth}_A(a), A] \cap U$ such that $f$ is constant on $\mathcal{AR}_{|a|+n}[a, B]$;

• [14] $U$ is weakly selective if for every $A \in U$ and every $\{A_b\}_{b \in \mathcal{AR}_1} \subseteq U|A$ with $[b, A_b] \neq \emptyset$ for each $b \in \mathcal{AR}_1[\emptyset, A]$ there exists $B \in U|A$ such that $[b, B] \subseteq [b, A_b]$ for every $b \in \mathcal{AR}_1[\emptyset, B]$;

In [14] Mijares showed that every Ramsey ultrafilter is weakly selective in a topological Ramsey space $\mathcal{R}$, provided that $\mathcal{R}$ satisfies an extra axiom, called (A8), as stated below.

(A8) For arbitrary $n \in \omega$, $A, B \in \mathcal{R}$ and $b \in \mathcal{AR}_n(B)$, if $\mathcal{AR}_{n+1}[b, B] \subseteq \mathcal{AR}_{n+1}[b, A]$ then $[b, B] \subseteq [b, A]$.

Interestingly, although many known examples of topological Ramsey spaces satisfy (A8), the spaces $\mathcal{E}_k (k \geq 2)$ are not so.

**Example.** Let $n = 1$. We give an example of $A, B \in \mathcal{E}_2$, $b \in \mathcal{AE}_2^1(B)$ with $r_2[b, B] \subseteq r_2[b, A]$ but $[b, B] \nsubseteq [b, A]$. Let

\[
\begin{align*}
b &= r_1(\omega^2) = \{(0, 0)\}, \\
A &= \{(2m, 2n) : 0 \leq m \leq n < \omega\}, \\
B &= \{(0, 2n) : 0 \leq n < \omega\} \cup \{(2m + 1, 2n + 1) : 0 \leq m \leq n < \omega\}.
\end{align*}
\]

The elements $A$ and $B$ are illustrated in Figures 3 and 4 with members of $A$ and $B$ shaded.

Unlike the case in the Ellentuck space and the Milliken space, the notion of ultrafilter on $\omega^{\ell k}$ generated by elements of the form $X \in \mathcal{E}_k$ does not correspond to that of ultrafilters in $\mathcal{E}_k$ (Definition 4.2). Suppose $\mathcal{V}$ is an ultrafilter in $\mathcal{E}_k$. We can find an ultrafilter $\mathcal{U}$ on $\omega^{\ell k}$ generated by $\mathcal{V}$. But conversely, it is not necessarily the case: Given an ultrafilter $\mathcal{U}$ on $\omega^{\ell k}$ generated by elements of the form $X \in \mathcal{E}_k$, let $\mathcal{V} = \mathcal{U} \cap \mathcal{E}_k$. We check that $\mathcal{V}$ does not always satisfy (a) (ii) in Definition 4.2, that is, even if $A, B \in \mathcal{V}$ and $a \in \mathcal{AE}_k$ are such that $[a, A]$ and $[a, B]$ are nonempty, we may not have $C \in \mathcal{F}$ with $C \in [a, A] \cap [a, B]$ as the following example shows.

Table 3: Element $A$ in Example ??
Example. Let $U$ be an ultrafilter on $\omega^2$ generated by elements of the form $X \in E_2$. Since

$$\omega^2 = \{(m, 2n) : 0 \leq m \leq 2n < \omega\} \cup \{(m, 2n_1) : 0 \leq m \leq 2n + 1 < \omega\},$$

we may assume without loss of generality that

$$A = \{(m, 2n) : 0 \leq m \leq 2n < \omega\}$$

is in $U$. Let

$$B = A \setminus \{(0, 2n) : 1 < n < \omega\} \cup \{(0, 2n - 1) : 1 < n < \omega\},$$

and

$$a = \{(0, 0)\}.$$

Note $A, B \in E_2$. Also, as it does not include any element of $E_2$, the set $\{(0, 2n) : 1 < n < \omega\}$ is not in $U$. Hence $B \in U$. The upper triangular representation of $A$ and $B$ are shown in Tables 5 and 6. Clearly, $[a, A]$ and $[a, B]$ are nonempty. However, any $C \in [a, A]$ must have infinitely many members of the form $(0, k) (k < \omega)$ from $A$, and cannot be an element of $[a, B]$.

Table 5: Element $A$ in Example ??

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Table 6: Element $B$ in Example ??

In the rest of this section, we show that the generic filter $B_k$ for $(E_k, \subseteq \text{FIN}^{=k})$ is a Nash-Williams ultrafilter, and it remains as a Nash-Williams ultrafilter after countable-support side-by-side Sacks forcing.
It is straightforward to show that $B_k$ is an ultrafilter on $\mathcal{W}_k$ in the extended model forced by $(\mathcal{E}_k, \subseteq^{\text{FIN}^\omega_k})$. We prove first that $B_k$ is a Nash-Williams ultrafilter.

**Lemma 4.4.** Let $G \subseteq \mathcal{A}\mathcal{E}_k$ be a Nash-Williams family and $G = G_0 \sqcup G_1$ be a partition. Then there exists $X \in B_k$ and $i \in 2$ such that $G_i|X = \emptyset$.

**Proof.** Given $G = G_0 \sqcup G_1$ as in the lemma, let

$$D_G = \{ X \in \mathcal{E}_k : (\exists i \in 2)(G_i|X = \emptyset) \}.$$

We check that $D_G$ is dense in $(\mathcal{E}_k, \subseteq^{\text{FIN}^\omega_k})$. By the Abstract Nash-Williams Theorem 2.8, for every $X \in \mathcal{E}_k$ there exists $Y \subseteq^{\text{FIN}^\omega_k} X$ such that $Y \in D_G$. Then $D_G$ is dense. By genericity of $B_k$, there exists $X \in B_k \cap D_G$, so $X \in B_k$ and $G_i|X = \emptyset$ for some $i \in 2$.

Now we check that $B_k$ is preserved as a Nash-Williams ultrafilter after countable-support side-by-side Sacks forcing. As in [22], this is done by proving that $B_k$ is localizing, i.e. the following theorem holds.

**Theorem 4.5** (local Parametrized High-dimensional Ellentuck Theorem). For every finite Souslin-measurable colouring of $\mathcal{E}_k \times \mathbb{R}^\omega$ there exists $B \in B_k$ and $p \in \mathcal{P}_\kappa$ such that $[\emptyset, B] \times [p]$ is monochromatic.

**Proof.** Fix a Souslin-measurable set $X \subseteq \mathcal{E}_k \times \mathbb{R}^\omega$. By Theorem 0.3, for every $X \in \mathcal{E}_k$ there exists $Y \subseteq X$ (so $Y \subseteq^{\text{FIN}^\omega_k} X$) and $p \in \mathcal{P}_\kappa$ such that $[\emptyset, Y] \times [p] \subseteq X$ or $[\emptyset, Y] \times [p] \cap X = \emptyset$. Therefore, the set

$$D_X = \{ Y \in \mathcal{E}_k : (\exists p \in \mathcal{P}_\kappa)(([\emptyset, Y] \times [p] \subseteq X) \lor ([\emptyset, Y] \times [p] \cap X = \emptyset)) \}$$

is dense in $(\mathcal{E}_k, \subseteq^{\text{FIN}^\omega_k})$. Hence by the genericity of $B_k$ we can find $B \in B_k \cap D_X$ satisfying the theorem.

Let $\kappa$ be an arbitrary cardinal and $\mathcal{P}_\kappa$ be the corresponding countable-support side-by-side Sacks forcing. The proofs of the lemmas below use Theorem 4.5 and closely follow those of [22, Thm. 3.4] and [21, Thm. 4.5], respectively.
Lemma 4.6. If $p \in P_\kappa$ and $p \Vdash \tau \subseteq W_k$ then there exists $q \leq p$ and $X \in B_k$ such that $q \Vdash X \subseteq \tau$ or $q \Vdash X \cap \tau = \emptyset$.

It follows that the upward closure of $B_k$ after forcing by $P_\kappa$ is still an ultrafilter.

Lemma 4.7. Suppose $p \in P_\kappa$ and $p \Vdash ((G \subseteq \mathcal{AE}_k \text{ is Nash-Williams}) \land (G = G_0 \cup G_1))$. Then there exists $q \leq p$, $X \in B_k$ and $i \in 2$ such that $q \Vdash F_i|X| = \emptyset$.

Therefore we have the following theorem.

Theorem 0.4. Let $k \in \omega$, $\kappa$ be an infinite cardinal and $P_\kappa$ be countable-support side-by-side Sacks forcing adding $\kappa$ Sacks reals. Let $B_k$ be a generic filter for $(\mathcal{E}_k, \subseteq_{\text{FIN}^*})$, and $\dot{V}$ a name for the upward closure $\{Y : (\exists X \in B_k)(X \leq Y)\}$ of $B_k$. Then $\Vdash p_\kappa (\dot{V} \text{ is a Nash-Williams ultrafilter in } \mathcal{E}_k)$.

In this paper, we proved a version of the Parametrized Ellentuck Theorem which is only moderately abstract – we request the topological Ramsey space to satisfy an extra condition $(L4)$. It would be interesting to know if the result still holds without $(L4)$ or with a weaker assumption than $(L4)$. We have also seen that the High-dimensional Ellentuck spaces are in some sense counter-intuitive – our usual methods of showing the relations among ultrafilters fail in these spaces. It would also be interesting to see how the exact relations among ultrafilters can be determined in the High-dimensional Ellentuck spaces, and in abstract topological Ramsey spaces.

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