

# GEOMETRY AND DYNAMICS ON SUBLINEARLY MORSE BOUNDARIES OF CAT(0) GROUPS

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ABSTRACT. Given a sublinear function  $\kappa$ ,  $\kappa$ -Morse boundaries  $\partial_\kappa X$  of proper CAT(0) spaces are introduced by Qing, Rafi and Tiozzo. It is a topological space consists of a large set of quasi-geodesic rays and it is quasi-isometrically invariant and metrizable. In this paper, we study the sublinearly Morse boundaries with the assumption that there is a cocompact action of a group  $G$  on the CAT(0) space in question. We show that  $G$  acts minimally on  $\partial_\kappa G$ . We also show that contracting elements of  $G$  induces a weak north-south dynamic on  $\partial_\kappa G$ . Furthermore, we show that a homeomorphism  $f: \partial_\kappa G \rightarrow \partial_\kappa G'$  comes from a quasi-isometry if and only if  $f$  is successively quasi-möbius and stable. Lastly, we characterize exactly when the sublinearly Morse boundary of a CAT(0) space is compact.

## 1. INTRODUCTION

Much of the geometric group theory originates from the studying of hyperbolic groups and hyperbolic spaces. Hyperbolic groups have solvable word problem and strong dynamical properties. One fundamental technique in the study of hyperbolic groups is to study the Gromov boundaries of these groups. Gromov took the collection of all infinite geodesic rays (up to fellow traveling) in the associated Cayley graph, equipped this set with cone topology, and defined the space to be the boundary  $\partial G$  of the hyperbolic group  $G$ . The boundary  $\partial G$  is independent of the choice of a generating set and has rich geometric, topological, and algebraic structures (see for example the survey by Kapovich and Benakli [KB02]).

If we view Gromov hyperbolic spaces as coarsely negatively curved, then CAT(0) spaces extends the space studied to include spaces with both local and global non-positive curvature. Accordingly the extension of the boundary theory to CAT(0) spaces and groups has also been developing in recent decades. In this setting, the space of all geodesic rays together with the cone topology is called the visual boundary (denoted by  $\partial_v X$ ). It is shown by Croke and Kleiner that the visual boundary of a CAT(0) space is not in general a quasi-isometry invariant [CK00]. In [Qin16], it is shown that, in the Croke-Kleiner example, failure to obtain quasi-isometry invariance can come from geodesic rays that spend linear amount of time (with respect to total time travelled) in each product region.

Hence, one can consider geodesic rays that spend a sublinear amount of time in each product region. In [QRT22a], Qing and Rafi introduce the sublinearly Morse boundary  $\partial_\kappa X$  of a CAT(0) metric space  $X$  and show that  $\partial_\kappa X$  is quasi-isometry invariant and metrizable. In [QRT22a], Qing and Tiozzo show that, for a right-angled Artin group  $G$ ,  $\partial_\kappa G$  is a model for Poisson boundaries associated to a random walk  $(G, \mu)$ . Intuitively, a (quasi-)geodesic ray is sublinearly Morse if it spends a sublinear amount of time in each maximal product region, with respect to

total time travelled when it enters that product region. Furthermore, in [GQR], it is shown that for every CAT(0) group with a rank one element, there exists a  $\kappa$  such that  $\partial_\kappa G$  can be identified with the Poisson boundaries of the group. The authors of [GQR] also show that the sublinearly Morse directions in the visual boundary of a rank 1 CAT(0) space with a geometric group action are generic with respect to Patterson-Sullivan measures. These are evidences that the sublinearly Morse directions constitutes a "large" subset of all directions. In this paper, we focus on the dynamical property of the group action on  $\kappa$ -Morse boundaries. Much of the work in this paper is inspired by the methods in [CM19] and [CS15]. In more general proper geodesic spaces, sublinearly Morse boundaries have been developed and studied in [QRT22b, MQZ22, QN22, QP22, IMZ20, DZ20], for instance.

*Minimality of the group action.* A group is said to act *minimally* on a topological space if every orbit is a dense subset of the space. We show that this property is enjoyed by the  $\kappa$  boundaries. In contrast with the identifications with Poisson boundaries in various settings, the minimality result evidence the fact that the boundary is not too large in excess of the orbit of a point under the group action.

**Theorem A.** (Theorem 3.6) *Suppose  $G$  is a group that acts geometrically on a CAT(0) space  $X$ . Then  $G$  acts minimally on  $\partial_\kappa G$ .*

Based on this result, we illustrate that for a subset of the group elements, their actions induces a north-south dynamics on the boundary:

**Theorem B.** (Corollary 3.8) *Suppose  $G$  is a group that acts geometrically on a CAT(0) space  $X$ . Let  $g \in G$  be a contracting element. For every open set  $V$  containing  $g^\infty$  and every compact set  $C \in (\partial_\kappa G \setminus [g^\infty])$ , there exists an  $N$  such that for all  $n \geq N$ , we have  $g^n C \subset V$ .*

*Compact type  $\kappa$ -boundaries.* In the examples shown in [QRT22a], the boundaries are not compact. We show that when  $X$  is a proper hyperbolic space,  $\kappa$ -boundary is homeomorphic to the associated Gromov boundary. In fact, we show that this is exactly when a cocompact CAT(0) space  $X$  has compact sublinear boundaries. On the other hand, examples of CAT(0) space without a cocompact group action, whose sublinearly boundaries are compact can be constructed easily. However it remains open to find a CAT(0) space  $X$  with non-compact sublinear boundary where  $\partial_\kappa X$  is a perfect space.

**Theorem C.** (Theorem 4.4) *Suppose a group  $G$  acts geometrically on a proper CAT(0) space  $X$  such that  $\partial_\kappa X \neq \emptyset$ , then the following are equivalent:*

- (1) *Every geodesic ray in  $X$  is  $\kappa$ -contracting.*
- (2) *Every geodesic ray in  $X$  is strongly contracting.*
- (3)  *$\partial_\kappa X$  is compact.*
- (4) *The space  $X$  is hyperbolic.*

**Corollary D.** *If  $X$  is a proper hyperbolic space then  $\partial_\kappa X \simeq \partial X$ .*

*Rigidity.* In 1996, Paulin gives the following characterization [Pau96]: if  $f : \partial X \rightarrow \partial Y$  is a homeomorphism between the boundaries of two proper, cocompact hyperbolic spaces, then the following are equivalent

- (1)  $f$  is induced by a quasi-isometry  $h : X \rightarrow Y$ .
- (2)  $f$  is quasi-möbius.

Quasi-möbius maps are maps such that changes in the cross ratio are controlled by a continuous function. We aim to give a similar characterization for sublinearly Morse boundaries. We use the notion of *successively quasi-möbius* discussed in [QRT22a], which is a 1-parameter family of quasi-möbius maps on  $\partial_\kappa X$ .

**Theorem E.** (*Theorem 5.1*) *Let  $X, Y$  be proper cocompact CAT(0) spaces with at least 3 points in their sublinear boundaries. A homeomorphism  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  is induced by a quasi-isometry  $h : X \rightarrow Y$  if and only if  $f$  is stable and successively quasi-möbius.*

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## 2. PRELIMINARIES

### 2.1. Quasi-isometry and quasi-isometric embeddings.

**Definition 2.1** (Quasi Isometric embedding). Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. For constants  $k \geq 1$  and  $K \geq 0$ , we say a map  $f : X \rightarrow Y$  is a  $(k, K)$ -quasi-isometric embedding if, for all points  $x_1, x_2 \in X$

$$\frac{1}{k}d_X(x_1, x_2) - K \leq d_Y(f(x_1), f(x_2)) \leq kd_X(x_1, x_2) + K.$$

If, in addition, every point in  $Y$  lies in the  $K$ -neighbourhood of the image of  $f$ , then  $f$  is called a  $(k, K)$ -quasi-isometry. When such a map exists,  $X$  and  $Y$  are said to be *quasi-isometric*.

A quasi-isometric embedding  $f^{-1} : Y \rightarrow X$  is called a *quasi-inverse* of  $f$  if for every  $x \in X$ ,  $d_X(x, f^{-1}f(x))$  is uniformly bounded above. In fact, after replacing  $k$  and  $K$  with larger constants, we assume that  $f^{-1}$  is also a  $(k, K)$ -quasi-isometric embedding,

$$\forall x \in X \quad d_X(x, f^{-1}f(x)) \leq K \quad \text{and} \quad \forall y \in Y \quad d_Y(y, ff^{-1}(y)) \leq K.$$

A *geodesic ray* in  $X$  is an isometric embedding  $\beta : [0, \infty) \rightarrow X$ . We fix a base-point  $\mathfrak{o} \in X$  and always assume that  $\beta(0) = \mathfrak{o}$ , that is, a geodesic ray is always assumed to start from this fixed base-point.

**Definition 2.2** (Quasi-geodesics). In this paper, a *quasi-geodesic ray* is a continuous quasi-isometric embedding  $\beta : [0, \infty) \rightarrow X$  starting from the basepoint  $\mathfrak{o}$ .

The additional assumption that quasi-geodesics are continuous is not necessary for the results in this paper to hold, but it is added for convenience and to make the exposition simpler.

If  $\beta : [0, \infty) \rightarrow X$  is a  $(q, Q)$ -quasi-isometric embedding, and  $f : X \rightarrow Y$  is a  $(k, K)$ -quasi-isometry then the composition  $f \circ \beta : [t_1, t_2] \rightarrow Y$  is a quasi-isometric embedding, but it may not be continuous. However, one can adjust the map slightly to make it continuous (see Definition 2.2 [QRT22a]) such that  $f \circ \beta$  is a  $(kq, 2(kq + kQ + K))$ -quasi-geodesic ray.

Similar to above, a *geodesic segment* is an isometric embedding  $\beta : [t_1, t_2] \rightarrow X$  and a *quasi-geodesic segment* is a continuous quasi-isometric embedding

$$\beta : [t_1, t_2] \rightarrow X.$$

**Notation.** In this paper we will use  $\alpha, \beta, \dots$  to denote quasi-geodesic rays. If the quasi-geodesic constants are  $(1, 0)$ , we use  $\alpha_0, \beta_0, \dots$  to signify that they are in

fact geodesic rays. Meanwhile, we use  $[\alpha], [\beta], \dots$  to denote equivalence classes of quasi-geodesic rays, and we also use  $\mathbf{a}, \mathbf{b}, \dots$  to denote equivalence classes without referring an element in each class. Furthermore, let  $\alpha$  be a (quasi-)geodesic ray  $\alpha: [0, \infty) \rightarrow X$ , if  $x_1, x_2$  are points on  $\alpha$ , then the segment of  $\alpha$  between  $x_1$  and  $x_2$  is denoted  $[x_1, x_2]_\alpha$ . If a segment is presented without subscript, for example  $[y_1, y_2]$ , then it is a geodesic segment between the two points. Let  $\beta$  be a quasi-geodesic ray. For  $r > 0$ , let  $t_r$  be the first time where  $\|\beta(t)\| = r$  and define:

$$(1) \quad \beta_r := \beta(t_r) \quad \text{and} \quad \beta|_r := \beta[0, t_r] = [\beta(0), \beta_r]_\beta$$

which are points and segments in  $X$ , respectively.

**2.2. Properties of CAT(0) spaces.** A geodesic metric space  $(X, d_X)$  is CAT(0) if geodesic triangles in  $X$  are at least as thin as triangles in Euclidean space with the same triple of side-lengths. To be precise, for any given geodesic triangle  $\Delta pqr$ , consider the unique triangle  $\Delta \bar{p}\bar{q}\bar{r}$  in the Euclidean plane with the same triple of side-lengths. The triangle  $\Delta pqr$  is at least as thin as  $\Delta \bar{p}\bar{q}\bar{r}$  in the following sense: For any pair of points  $x, y$  on the triangle  $\Delta pqr$ , without loss of generality let  $x, y$  be on edges  $[p, q]$  and  $[p, r]$ , if we choose points  $\bar{x}$  and  $\bar{y}$  on edges  $[\bar{p}, \bar{q}]$  and  $[\bar{p}, \bar{r}]$  of the triangle  $\Delta \bar{p}\bar{q}\bar{r}$  so that  $d_X(p, x) = d(\bar{p}, \bar{x})$  and  $d_X(p, y) = d(\bar{p}, \bar{y})$  then,

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

A metric space  $X$  is *proper* if closed metric balls are compact. For the remainder of the paper, we assume  $X$  is a proper CAT(0) space; a proper CAT(0) space has the following basic properties that are needed in this paper:

**Lemma 2.3.** *A proper CAT(0) space  $X$  has the following properties:*

- (1) *For any two points  $x, y$  in  $X$ , there exists exactly one geodesic connecting them. Consequently,  $X$  is contractible via geodesic retraction to a base point in the space.*
- (2) *The nearest point projection from a point  $x$  to a geodesic line  $\beta_0$  is a unique point denoted  $\pi_{\beta_0}(x)$ , or simply  $x_{\beta_0}$ . In fact, the closest point projection map to a geodesic*

$$\pi_{\beta_0}: X \rightarrow \beta_0$$

*is Lipschitz with respect to distances. The nearest point projection from a point  $x$  to a quasi-geodesic line  $\beta$  exists and is not necessarily unique. We denote the whole projection set  $\pi_\beta(x)$ .*

- (3) *For any  $x \in X$ , the distance function  $d_X(x, \cdot)$  is convex. In other words, for any given any geodesic  $[x_0, x_1]$  and  $t \in [0, 1]$ , if  $x_t$  satisfies  $d_X(x_0, x_t) = td(x_0, x_1)$  then we must have*

$$d_X(x, x_t) \leq (1-t)d_X(x, x_0) + td_X(x, x_1).$$

In addition, we need the following redirecting surgery for all proper metric spaces:

**Lemma 2.4.** *Let  $X$  be a proper, complete metric space. Let  $b$  be a geodesic ray and  $\gamma$  be a  $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray. For  $r > 0$ , assume that  $d_X(b_r, \gamma) \leq r/2$ . Then, there exists a  $(9\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic  $\gamma'$  so that*

$$\gamma' \in [b], \quad \text{and} \quad \gamma|_{r/2} = \gamma'|_{r/2}.$$

**2.3. Boundaries of CAT(0) space.**

2.3.1. *Visual boundary.* In this section we review a couple of topological boundaries of CAT(0) spaces that are important to the study of this paper.

**Definition 2.5** (visual boundary). Let  $X$  be a CAT(0) space. The *visual boundary* of  $X$ , denoted  $\partial_v X$ , is the collection of equivalence classes of infinite geodesic rays, where  $\alpha$  and  $\beta$  are in the same equivalence class, if and only if there exists some  $C \geq 0$  such that  $d(\alpha(t), \beta(t)) \leq C$  for all  $t \in [0, \infty)$ . The equivalence class of  $\alpha$  in  $\partial_v X$  we denote  $\alpha(\infty)$ .

Notice that by Proposition I. 8.2 in [BH99], for each  $\alpha$  representing an element of  $\partial X$ , and for each  $x' \in X$ , there is a unique geodesic ray  $\alpha'$  starting at  $x'$  with  $\alpha(\infty) = \alpha'(\infty)$ .

We describe the topology of the visual boundary by a neighbourhood basis: fix a base point  $\mathfrak{o}$  and let  $\alpha$  be a geodesic ray starting at  $\mathfrak{o}$ . A neighborhood basis for  $\alpha$  is given by sets of the form:

$$\mathcal{U}_v(\alpha(\infty), r, \epsilon) := \{\beta(\infty) \in \partial_v X \mid \beta(0) = \mathfrak{o} \text{ and } d(\alpha(t), \beta(t)) < \epsilon \text{ for all } t < r\}.$$

In other words, two geodesic rays are close if they have geodesic representatives that start at the same point and stay close (are at most  $\epsilon$  apart) for a long time (at least  $r$ ). Notice that the above definition of the topology on  $\partial_v X$  references a base-point  $\mathfrak{o}$ . Nonetheless, Proposition I. 8.8 in [BH99] proves that the topology of the visual boundary is base-point invariant.

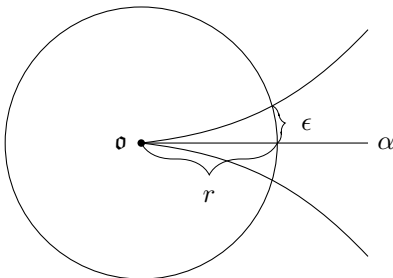


FIGURE 1. A basis for open sets

2.3.2. *Sublinearly Morse boundaries.*

Let  $\kappa: [0, \infty) \rightarrow [1, \infty)$  be a sublinear function that is monotone increasing and concave. That is

$$\lim_{t \rightarrow \infty} \frac{\kappa(t)}{t} = 0.$$

The assumption that  $\kappa$  is increasing and concave makes certain arguments cleaner, otherwise they are not really needed. One can always replace any sub-linear function  $\kappa$ , with another sub-linear function  $\bar{\kappa}$  so that

$$\kappa(t) \leq \bar{\kappa}(t) \leq C \kappa(t)$$

for some constant  $C$  and  $\bar{\kappa}$  is monotone increasing and concave. For example, define

$$\bar{\kappa}(t) = \sup \left\{ \lambda \kappa(u) + (1 - \lambda) \kappa(v) \mid 0 \leq \lambda \leq 1, u, v > 0, \text{ and } \lambda u + (1 - \lambda)v = t \right\}.$$

The requirement  $\kappa(t) \geq 1$  is there to remove additive errors in the definition of  $\kappa$ -contracting geodesics(See Definition 2.7).

2.3.3.  $\kappa$ -Morse geodesic rays. The boundary of interest in this paper consists of points in  $\partial X$  that are in the “hyperbolic-like”. In proper CAT(0) spaces, they can be characterized in two equivalence ways.

**Definition 2.6** ( $\kappa$ -neighborhood,  $\kappa$ -Morse,  $\kappa$ -weakly Morse). For a closed set  $Z$  and a constant  $n$  define the  $(\kappa, n)$ -neighbourhood of  $Z$  to be

$$\mathcal{N}_\kappa(Z, n) = \left\{ x \in X \mid d_X(x, Z) \leq n \cdot \kappa(x) \right\}.$$

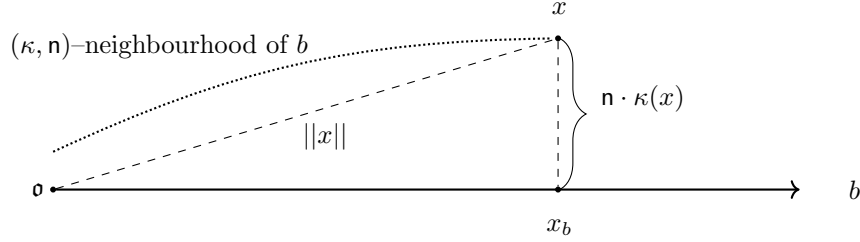


FIGURE 2. A  $\kappa$ -neighbourhood of a geodesic ray  $b$  with multiplicative constant  $n$ .

Let  $Z \subseteq X$  be a closed set, and let  $\kappa$  be a concave sublinear function. We say that  $Z$  is  $\kappa$ -Morse if there exists a proper function  $m_Z : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for any sublinear function  $\kappa'$  and for any  $r > 0$ , there exists  $R$  such that for any  $(\mathbf{q}, \mathbf{Q})$ -quasi-geodesic ray  $\beta$  with  $m_Z(\mathbf{q}, \mathbf{Q})$  small compared to  $r$ , if

$$d_X(\beta_R, Z) \leq \kappa'(R) \quad \text{then} \quad \beta|_r \subset \mathcal{N}_\kappa(Z, m_Z(\mathbf{q}, \mathbf{Q}))$$

The function  $m_Z$  will be called a *Morse gauge* of  $Z$ .

We say a closed subset  $Z$  of  $X$  is  $\kappa$ -weakly Morse if there is a function

$$m_Z : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$$

so that if  $\beta : [s, t] \rightarrow X$  is a  $(\mathbf{q}, \mathbf{Q})$ -quasi-geodesic with end points on  $Z$  then

$$[s, t]_\beta \subset \mathcal{N}_\kappa(Z, m_Z(\mathbf{q}, \mathbf{Q})).$$

We refer to  $m_Z$  as the *Morse gauge* for  $Z$ . We always assume

$$(2) \quad m_Z(\mathbf{q}, \mathbf{Q}) \geq \max(\mathbf{q}, \mathbf{Q}).$$

**Definition 2.7** ( $\kappa$ -contracting sets). For  $x \in X$ , define  $\|x\| = d_X(o, x)$ . For a closed subspace  $Z$  of  $X$ , we say  $Z$  is  $\kappa$ -contracting if there is a constant  $c_Z$  so that, for every  $x, y \in X$

$$d_X(x, y) \leq d_X(x, Z) \implies \text{diam}_X(x_Z \cup y_Z) \leq c_Z \cdot \kappa(\|x\|).$$

In fact, to simplify notation, we drop  $\|\cdot\|$  when it appears in the  $\kappa$  function and write  $\kappa(x)$  instead of  $\kappa(\|x\|)$ .

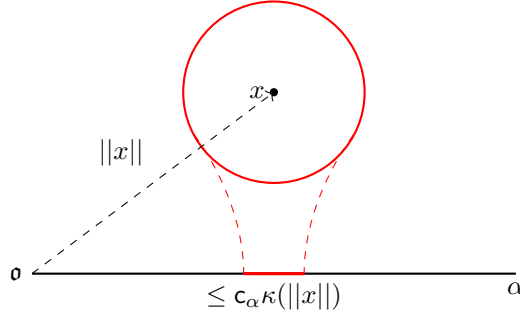


FIGURE 3. A  $\kappa$ -contracting geodesic ray.

In CAT(0) spaces, a geodesic is  $\kappa$ -contracting if and only if it is  $\kappa$ -Morse [QRT22a]. In fact, the following technical tool is used several times in this paper. It states that if a geodesic ray  $Z$  is  $\kappa$ -contracting, then it is  $\kappa$ -strongly Morse. Compared with Definition 2.6,  $\kappa$ -strongly Morse only requires the endpoint of the quasi-geodesic segment to be sublinearly close to  $Z$  instead of landing exactly on  $Z$ .

Quasi-geodesic rays in  $X$  are grouped into equivalence classes to form  $\partial_\kappa X$ .

**Definition 2.8** ( $\kappa$ -equivalence classes in  $\partial_\kappa X$ ). Let  $\beta$  and  $\gamma$  be two quasi-geodesic rays in  $X$ . If  $\beta$  is in some  $\kappa$ -neighbourhood of  $\gamma$  and  $\gamma$  is in some  $\kappa$ -neighbourhood of  $\beta$ , we say that  $\beta$  and  $\gamma$   $\kappa$ -fellow travel each other. This defines an equivalence relation on the set of quasi-geodesic rays in  $X$  (to obtain transitivity, one needs to change  $n$  of the associated  $(\kappa, n)$ -neighbourhood).

We denote the equivalence class that contains  $\beta$  by  $[\beta]$ :

**Definition 2.9** (Sublinearly Morse boundary). Let  $\kappa$  be a sublinear function as specified in Section 2.3.2 and let  $X$  be a CAT(0) space.

$$\partial_\kappa X := \{ \text{all } \kappa\text{-Morse quasi-geodesics} \} / \kappa\text{-fellow travelling}$$

We define the topology of  $\partial_\kappa X$  in Section 2.3.4.

We also use  $\mathbf{a}, \mathbf{b}$  to denote  $\kappa$ -equivalence classes in  $\partial_\kappa X$ . We need the following fact that since  $X$  is CAT(0), there is a unique geodesic ray in each equivalence class:

**Lemma 2.10** (Lemma 3.5, [QRT22a]). *Let  $X$  be a CAT(0) space. Let  $b: [0, \infty) \rightarrow X$  be a geodesic ray in  $X$ . Then  $b$  is the unique geodesic ray in any  $(\kappa, n)$ -neighbourhood of  $b$  for any  $n$ . That is to say, there is an 1-1 embedding of the set of points in  $\partial_\kappa X$  into the points of  $\partial_v X$ .*

*Proof.* For each element  $\mathbf{a} \in \partial_\kappa X$ , consider its unique geodesic ray  $\alpha$ . The associated  $\alpha(\infty)$  is a element of  $\partial_v X$ . By Lemma 3.5, [QRT22a], each equivalence class contains a unique geodesic ray. Meanwhile, if two elements  $\mathbf{a}, \mathbf{b} \in \partial_\kappa X$  contain the same geodesic ray, they are in fact the same set of quasi-geodesics, therefore this map is well-defined. Therefore we have an embedding of the set of points in  $\partial_\kappa X$  into the points of  $\partial_v X$ .  $\square$

2.3.4. *Coarse cone topology on  $\partial_\kappa X$ .* We equip  $\partial_\kappa X$  with a topology which is a coarse version of the visual topology. In visual topology, if two geodesic rays follow travel for a long time, then they are “close”. In this coarse version, if two geodesic rays and all the quasi-geodesic rays in their respect equivalence classes remain close for a long time, then they are close. Now we define it formally. First, we say a quantity  $D$  is small compared to a radius  $r > 0$  if

$$(3) \quad D \leq \frac{r}{2\kappa(r)}.$$

Recall that given a  $\kappa$ -Morse quasi-geodesic ray  $\beta$ , we denotes its associated Morse gauges functions  $\mathfrak{m}_\beta(\mathfrak{q}, \mathfrak{Q})$ . These are multiplicative constants that give the heights of the  $\kappa$ -neighbourhoods.

**Definition 2.11** (topology on  $\partial_\kappa X$ ). Let  $\mathfrak{a} \in \partial_\kappa X$  and  $\alpha_0 \in \mathfrak{a}$  be the unique geodesic in the class  $\mathfrak{a}$ . Define  $\mathcal{U}_\kappa(\mathfrak{a}, r)$  to be the set of points  $\mathfrak{b}$  such that for any  $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic of  $\mathfrak{b}$ , denoted  $\beta$ , such that  $\mathfrak{m}_\beta(\mathfrak{q}, \mathfrak{Q})$  is small compared to  $r$ , satisfies

$$\beta|_r \subset \mathcal{N}_\kappa(\alpha_0, \mathfrak{m}_{\alpha_0}(\mathfrak{q}, \mathfrak{Q})).$$

Let the topology of  $\partial_\kappa X$  be the topology induced by this neighbourhood system. The following fact shows that a  $\kappa$ -boundary is well defined with respect the associated group.

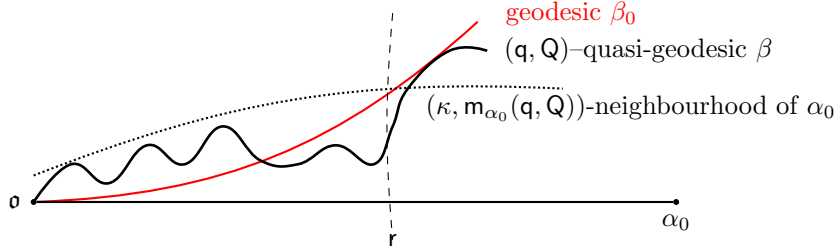


FIGURE 4.  $\mathfrak{b} \in \mathcal{U}_\kappa(\mathfrak{a}, r)$  because the quasi-geodesics of  $\mathfrak{b}$  such as  $\beta, \beta_0$  stay inside the associated  $(\kappa, \mathfrak{m}_{\alpha_0}(\mathfrak{q}, \mathfrak{Q}))$ -neighborhood of  $\alpha_0$  (as in Definition 2.6 ), up to distance  $r$ .

**Theorem 2.12** ([QRT22b]). *Let  $X$  be a proper metric space and let  $\kappa$  be a sublinear function. The  $\kappa$ -boundary of  $X$ , denoted  $\partial_\kappa X$ , is a quasi-isometrically invariant topological space. Furthermore,  $\partial_\kappa X$  is metrizable.*

### 3. DENSE SUBSETS AND MINIMALITY OF $G$ -ACTION

In this section we prove two results concerning dense subsets of  $\partial_\kappa G$ . First we show that the set of all 1-Morse directions,  $\partial_1 G$  is dense in  $\partial_\kappa G$ , secondly and more generally, the action of  $G$  is minimal on  $\partial_\kappa G$  and as a consequence, a Morse element in  $G$  acts with North-South dynamics on the boundary. To begin with, in this section, let  $G$  acts geometrically on a CAT(0) space  $X$ .

**Definition 3.1.** We say that a word  $g \in G$  tracks a geodesic segment  $\alpha: [0, t] \rightarrow X$  starting at the base-point, if there exists a strict fundamental domain  $D$  of the  $G$ -action, and  $\alpha \subset g \cdot D$ .



We also recall that to notate a segment of a quasi-geodesic  $\alpha$  between two points  $x$  and  $y$ , we write  $[x, y]_\alpha$ , and when  $\alpha$  is a geodesic we sometimes drop the subscript and write  $[x, y]$ .

**Lemma 3.2.** [*QRT22b, Lemma 3.6*] Consider a quasi-geodesic segment  $[x, y]_\alpha \in X$  that is  $(q, Q)$ -quasi-geodesic and another  $(q, Q)$ -quasi-geodesic segment  $\beta$  connecting a point  $z \in X$  to a point  $w \in X$ . Let  $D > 0$ , assume that for every point  $u \in \alpha$ ,  $y$  is a point in  $N_D(u_\beta)$ , and let  $\gamma$  be the concatenation of the quasi-geodesic segment  $[x, y]_\alpha$  and the quasi-geodesic segment  $[y, z]_\beta$ . Then  $\gamma = [x, y]_\alpha \cup [y, z]_\beta$  is a  $(3q, 3Q + D)$ -quasi-geodesic.

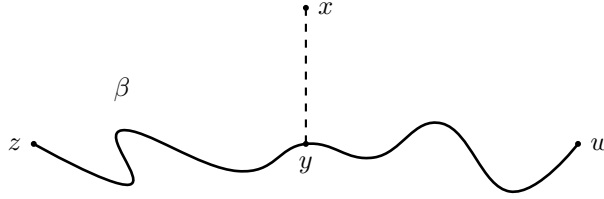


FIGURE 5. For  $y \in x_\beta$ , the concatenation of the quasi-geodesic segment  $[x, y]_\alpha$  and the quasi-geodesic segment  $[y, z]_\beta$  is a quasi-geodesic.

Next we show that  $\kappa$ -Morse elements have finite projections to each other.

**Lemma 3.3.** Let  $X$  be a proper metric space and let  $\alpha, \beta$  be quasi-geodesic rays in distinct elements of  $\partial_\kappa X$ , then the projection of  $\alpha$  to  $\beta$  is a finite set  $D$  that depends only on the quasi-geodesic constants of  $\alpha$  and  $\beta$ .

*Proof.* Consider the projection of a point  $\alpha(t)$  to  $\beta$  and name it  $y$ . By Lemma 3.2, the concatenation  $[\sigma, y]_\beta \cup [y, \alpha(t)]$  is a quasi-geodesic segment with endpoints on  $\alpha$  and with uniform constants. Since  $\alpha$  is  $\kappa$ -Morse, it is weakly  $\kappa$ -Morse (Proposition A.8, [QRT22b]), thus  $[\sigma, y]_\beta \cup [y, \alpha(t)]$  resides in a sublinear neighborhood of  $\alpha$ , label this neighbourhood  $K$ . If the projection of  $\alpha$  to  $\beta$  is infinite diameter, then arbitrarily long segment of  $\beta$  is in  $K$  and we have that  $\beta \sim \alpha$ , contrary to our assumption. Thus the projection of  $\alpha$  to  $\beta$  is finite diameter.  $\square$

**Lemma 3.4.** Let  $\alpha, \beta$  be  $(q, Q)$ -quasi-geodesic representatives in distinct elements of  $\partial_\kappa X$ , and let  $D > 0$ ,  $r_0 > 0$ , then there exists a quasi-geodesic ray  $\gamma$  and  $r > r_0$  such that

$$\gamma|_r = \alpha \text{ and } \gamma|_{R^+} = \beta|_{R^+}$$

and  $\gamma$  is a  $(3q, 3Q + D)$ -quasi-geodesic ray.

*Proof.* By Lemma 3.3, the projection of  $\alpha$  to  $\beta$  is bounded, given any  $D > 0$ , there exists a  $t_0$  such that the projection of  $\alpha[t_0, \infty)$  to  $\beta$  is bounded above by  $D$ . Take  $r_0 = \beta(t_0)$ . Therefore, by Lemma 3.2, we have that the quasi-geodesic ray  $\gamma_r$  redirects from  $\beta$  to  $\alpha$  after  $r$  is a  $(3q, 3Q + D)$ -quasi-geodesic ray.  $\square$

**Lemma 3.5.** Suppose  $c_0$  is a concatenation of an initial segment of a  $\kappa$ -Morse geodesic ray  $\underline{b}$  and another  $\kappa$ -Morse geodesic ray  $\underline{a}$ , and  $c_0$  is a  $(3, 0)$ -quasi-geodesic ray. Then  $c_0$  is also  $\kappa$ -Morse, and its Morse gauge is bounded above by

$$m_{c_0}(q, Q) \leq 2m_b(3q, Q) + 2m_a(3q, Q).$$

*Proof.* Consider a quasi-geodesic ray  $c'$  that sublinearly tracks  $c_0$ . Let  $p$  be the point of concatenation on  $c_0$  and project  $p$  to  $c'$ . By Lemma?? the first part of  $c_0$  is a  $(3q, Q)$  quasi-geodesic ray whose endpoints are on  $b$ ; the rest is part of a  $(3q, Q)$  quasi-geodesic ray that sublinearly tracks  $a$ . Thus  $c'$  is in a  $m_b(3q, Q) + m_a(3q, Q)$ -neighbourhood of  $c_0$ . (we can shift this gauge further to a reference geodesic but it may not be necessary. Also the Morse gauges with respect to different but equivalent definitions are off by a constant, see [?])  $\square$

**Theorem 3.6.** *Suppose  $G$  acts geometrically on a proper  $CAT(0)$  space  $X$  with  $|\partial_\kappa X| \geq 3$ . There exists  $\mathfrak{a} \in \partial_\kappa X$ , such that  $G \cdot \mathfrak{a}$  is dense in  $\partial_\kappa X$ .*

*Proof.* Fix a point  $\mathfrak{b} \in \partial_\kappa X$  and let  $b$  be a geodesic ray representing  $\mathfrak{b}$  with  $b(0) = \mathfrak{o}$ . Since  $\partial_\kappa X \neq \emptyset$ , the group  $G$  contains a rank one element  $g$  by [Zal22]. Let  $g^+, g^-$  represent the points in  $\partial_\kappa X$  which are defined by an axis of  $g$ . Notice since  $|\partial_\kappa X| \geq 3$ , we get that the visual boundary  $\partial X$  has cardinality at least 3 and in particular,  $G$  is not virtually  $\mathbb{Z}$ . Thus, using Corollary 4.11 of [Mur19], the group  $G$  can't fix  $g^+$ . Pick a point  $hg^+ \in G \cdot g^+$  with  $hg^+ \neq g^+$  and let  $l$  be a geodesic line connecting  $g^+$  to  $hg^+$ . This is possibly since the Morse and the sublinearly Morse boundaries are visibility spaces ( [Zal22], [CCM19], [Cor17]). Let  $p$  be a point on  $l$ , using cocompactness of the action, there exist a constant  $C$  and element  $g_i \in G$  with  $d(g_i p, b(i)) \leq C$  for all  $i \in \mathbb{N}$ . Denote  $l_i := g_i l$  and  $p_i := g_i p$  and consider the quasi-geodesics  $\alpha_i := [\mathfrak{o}, p_i] \cup [p_i, l_i(\infty))$ ,  $\beta_i := [\mathfrak{o}, p_i] \cup [p_i, l_i(-\infty))$ . Since  $l_i$  is a line in a  $CAT(0)$  space, at least one of the angles  $\angle(\mathfrak{o}, p_i, l_i(\infty))$ ,  $\angle(\mathfrak{o}, p_i, l_i(-\infty))$  is greater than or equal  $\frac{\pi}{2}$ . Hence, at least one of  $i$  or  $\beta_i$  is a  $(3, 0)$ -quasi-geodesic ray.

Now the sequence of  $(3, 0)$ -quasi-geodesic ray (as a subset of the  $\alpha_i$ 's and the  $\beta_i$ 's) we rename them  $\{\gamma_i, i = 1, 2, 3, \dots\}$ . Each  $\gamma_i$  is Morse as its tail is Morse. By Lemma 3.5, for all  $i$  and for all  $q, Q$  we have,

$$m_{\gamma_i}(q, Q) < m_b(3q, Q) + m_l(3q, Q) + C',$$

where  $C'$  is a constant depending only on  $C$ . In particular, there exists a Morse gauge  $m \geq m_b$  such that  $\gamma_i$  are all  $\kappa$ -Morse with gauge  $m$ . Let  $r$  be given and let  $q, Q$  be small compared to  $r$ . Since  $b$  is  $\kappa$ -Morse, it is strongly  $\kappa$ -Morse. Fix a sublinear function  $\kappa'$  to be determined later, this yields an  $R \geq 1$  such that the conclusion of the the strongly  $\kappa$ -Morse notion holds. Choose  $i$  large enough so that  $d(b(R), \gamma_i) \leq C$ . Thus, there exist a point  $s_i \in [0, \infty)$  with  $d(b(R), \gamma_i(s_i)) \leq C$ . In particular, since  $\gamma_i$  is a  $(3, 0)$ -quasi-geodesic, we have

$$R - D \leq d(\gamma(s_i), b(0)) \leq R + D,$$

for a constant  $D \geq 1$  depending only on  $C$ . Now, if  $\zeta'$  is a  $(q, Q)$ -quasi-geodesic in the class of  $\gamma_i$ , then, since  $\gamma_i$  is strongly  $\kappa$ -Morse, we get that  $d(\gamma_i(s_i), \zeta) \leq m\kappa(s_i) \leq m\kappa(R + D) \leq mD\kappa(R) \leq mD\kappa(R)$ , where the last two inequalities hold as  $D, R \geq 1$ . This provides a point  $x \in \zeta$  with  $d(x, \gamma_i(s_i)) \leq mD\kappa(R)$ . Hence, by the triangle inequality, we get that  $d(x, b(R)) \leq mD\kappa(R) + C$ . Now, choosing  $\kappa' = mD\kappa + C$  and using the fact that  $b$  is strongly  $\kappa$ -Morse, we get that  $\zeta_r \subseteq \mathcal{N}_\kappa(b, m_b)$  concluding the proof.

□

**Corollary 3.7.** *Every non-empty  $\partial_\kappa G$  has a rank one isometry, furthermore,  $\partial_\kappa G$  has a dense subset consists of all Morse geodesic rays.*

*Proof.* If  $\partial_\kappa X$  is non-empty, consider an element  $\mathbf{b} \in \partial_\kappa X$ . Since  $b_0$  is sublinearly Morse, it does not bound a half flat. Therefore there exists a segment on  $b_0$ :  $L = [\mathfrak{o}, x]_{b_0}$  such that  $L$  does not bound a half flat. Consider the word that tracks  $L$ , call it  $g_L$ .  $g_L$  is an axial isometry, whose axis overlaps with  $N_D(b_0)$  up till the point  $x$ . Therefore  $g_L$  does not bound a half flat either. Thus  $g_L$  is a rank one isometry.

Let  $g_L$  be the element  $\mathbf{a}$  in Theorem 3.6. we have that  $G \cdot [l_g]$  is dense in  $\partial_\kappa X$  as desired.

□

Now we prove a weak version of North-South dynamics for the action of a group on its  $\kappa$ -boundaries.

**Theorem 3.8.** *Let  $g \in G$  be a Morse element. For an open set  $V$  containing  $g^\infty$  and a compact set  $\mathcal{C} \in (\partial_\kappa G \setminus [g^{-\infty}])$ , there exists an  $N$  such that for all  $n \geq N$ , we have  $g^n \cdot \mathcal{C} \subset V$ .*

*Proof.* By visibility of the Morse boundary of CAT(0) spaces [CS15], there exists a geodesic line connecting  $g^\infty$  and  $g^{-\infty}$ . Furthermore, the geodesic line is the axis of  $g$  and hence  $g^n$  tracks longer and longer segments of this axis. Let  $\mathbf{c} \in \mathcal{C}$  be a  $\kappa$ -Morse element. Let  $r$  be given and let  $r_{g^+}$  be the radius provided by Proposition 4.4 in [QRT22a]. Arguing exactly as in Theorem 3.6, we get some integer  $n(\mathbf{c})$  depending only on  $r_{g^+}$  such that

$$g^{n(\mathbf{c})} \cdot \mathbf{c} \in \mathcal{U}([g^\infty], r_{g^+}).$$

Applying Proposition 4.4 [QRT22a], there exists  $R_{\mathbf{c}}$  such that

$$g^{n(\mathbf{c})} \cdot \mathcal{U}_\kappa(\mathbf{c}, R_{\mathbf{c}}) \subset \mathcal{U}_\kappa([g^\infty], r).$$

Since  $\{\mathcal{U}_\kappa(\mathbf{c}, R_{\mathbf{c}})\}_{\mathbf{c} \in \mathcal{C}}$  covers  $\mathcal{C}$  and  $\mathcal{C}$  is compact, we get a finite subcover

$$\{\mathcal{U}_\kappa(\mathbf{c}_i, R_{\mathbf{c}_i})\}_{i=1}^m.$$

Using the above, for each  $i \in \{1, \dots, m\}$ , we get an integer  $n(\mathbf{c}_i)$  with

$$g^{n(\mathbf{c}_i)} \cdot \mathcal{U}_\kappa(\mathbf{c}_i, R_{\mathbf{c}_i}) \subset \mathcal{U}_\kappa([g^\infty], r).$$

Choosing  $k = \max\{n(\mathbf{c}_i)\}_{i=1}^m$  yields that  $g^k \mathcal{C} \subseteq \mathcal{U}_\kappa([g^\infty], r)$  concluding the proof.

□

#### 4. COMPACT TYPE $\kappa$ -BOUNDARIES FOR CAT(0) SPACES

In [QRT22a], it is shown that if  $G = \mathbb{Z}^2 \star \mathbb{Z}$ , then  $\partial_\kappa G$  is not compact. In this section we show that the  $\kappa$ -boundary is compact if and only if the underlying space is hyperbolic and proper.

Consider the subset of all D-strongly contracting geodesic rays emanating from  $\mathfrak{o}$  in a CAT(0) space  $X$ . The following lemma states that equipping this subset with the subspace topology of the visual boundary or the subspace topology of the  $\kappa$ -boundary yields homeomorphic spaces. The intuitive reason for this is the following: since quasi-geodesics stay uniformly close to D-strongly contracting geodesics, the

topology of fellow travelling of geodesics (the visual topology) and the topology of fellow travelling of quasi-geodesics (the topology of the  $\kappa$ -boundary) coincide.

Recall that we use  $\partial_v^D X$  to denote the set of all D-contracting geodesic rays emanating from  $\mathfrak{o}$  when equipped with the subspace topology of the visual boundary, and use  $\partial_\kappa^D X$  when equipped with the subspace topology of the  $\kappa$ -boundary.

**Proposition 4.1.** *The identity map  $id : \partial_v^D X \rightarrow \partial_\kappa^D X$  is a homeomorphism.*

*Proof.* We need to show that the map  $id : \partial_v^D X \rightarrow \partial_\kappa^D X$  is a homeomorphism. Since  $\partial_v^D X$  is closed (Lemma 3.2 in [CS15]) and  $X$  is proper,  $\partial_v^D X$  must be compact. Also, the space  $\partial_\kappa^D X$  is metrizable by Theorem D in [QRT22a]. Hence, it suffices to show that the map  $id$  is a continuous map. Notice that since every geodesic ray in  $\partial_v^D X$  is D-strongly contracting for the same D, applying Definition 2.6, we get an associated Morse function such that every geodesic ray  $\beta_0$  is  $\mathfrak{m}(D)$ -Morse, where  $\mathfrak{m}(D)$  depends only on D and satisfies the following: For every constants  $r > 0$ ,  $n > 0$  and every sublinear function  $\kappa'$ , there is an  $R = R(\beta_0, r, n, \kappa') > 0$  where the following holds: Let  $\eta : [0, \infty) \rightarrow X$  be a  $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray so that  $\mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q})$  is small compared to  $r$ , let  $t_r$  be the first time  $\|\eta(t_r)\| = r$  and let  $t_R$  be the first time  $\|\eta(t_R)\| = R$ . Then

$$d_X(\eta(t_R), \beta_0) \leq n \cdot \kappa'(R) \implies \eta[0, t_r] \subset \mathcal{N}_1(\beta_0, \mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q})) \subset \mathcal{N}_\kappa(\beta_0, \mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q})).$$

We first claim the following:

**Claim.** Given  $\mathfrak{b} \in \partial_\kappa^D X$ , each neighbourhood of  $\mathfrak{b}$ , denoted  $\mathcal{U}_\kappa(\mathfrak{b}, r)$ , must contain a visual neighbourhood basis of  $\beta_0$ , the unique geodesic ray in the class of  $\mathfrak{b}$ .

*Proof.* To see this, let  $\beta_0 \in \mathfrak{b}$  be the unique geodesic ray starting at  $\mathfrak{o}$ . We wish to show that for any  $r > 0$ , there exists  $r'$  and  $\epsilon$  such that  $\mathcal{U}_v(\beta_0, r', \epsilon) \subseteq \mathcal{U}_\kappa(\mathfrak{b}, r)$

In other words, we want to show that for any  $r > 0$ , there exists  $r'$  and  $\epsilon$  if a geodesic ray  $\alpha_0 \in \mathfrak{a}$  with  $\alpha_0(0) = \mathfrak{o}$  satisfies  $d(\alpha_0(t), \beta_0(t)) < \epsilon$  for  $t \leq r'$ , then, any  $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic representative  $\alpha$  of  $\mathfrak{a}$  with  $\mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q})$  small compared to  $r$ , we have

$$\alpha|_r \subset \mathcal{N}_\kappa(\zeta, \mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q})).$$

Remember that  $\alpha|_r = \alpha([0, t_r])$  where  $t_r$  is the first time where  $\|\alpha(t)\| = r$ .

Let  $r$  be given and let

$$n = \max\{\mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q}) + 1 | \mathfrak{q}, \mathfrak{Q} \leq r\}.$$

By Definition 2.6, with  $Z = \beta_0$ , there exists an  $R = R(r, n)$  such that any  $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic representative  $\beta$  of  $\mathfrak{a}$  with  $\mathfrak{m}_\beta(\mathfrak{q}, \mathfrak{Q})$  small compared to  $r$ , we have

$$d(\beta(t_R), b) < n \implies \beta_r \subset \mathcal{N}_1(\beta_0, \mathfrak{m}_\beta(\mathfrak{q}, \mathfrak{Q})).$$

Choose  $r' = r + R$  and  $\epsilon = 1$ . Hence, we want to show that if  $d(a(t), b(t)) < 1$  for  $t \leq r + R$ , then  $\beta_r \subset \mathcal{N}_\kappa(b, \mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q}))$  for  $\beta$  defined above. Since  $a$  is 1-Morse with gauge  $\mathfrak{m}$ , the Hausdorff distance between  $a$  and  $\beta$  is at most  $\mathfrak{m}(\mathfrak{q}, \mathfrak{Q})$ . This implies that for any  $0 < t \leq r + R$ , we have

$$d(a(t), \beta(i_t)) < \mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q}),$$

for some  $i_t$ .

Therefore, if  $t_R$  is the first time with  $\|\beta(t_R)\| = R$ , we must have

$$d(a, \beta(t_R)) < \mathfrak{m}_{\beta_0}(\mathfrak{q}, \mathfrak{Q}).$$

Now, since  $d(a(t), b(t)) < 1$  for all  $t < r + R$  and as  $d(a, \beta(t_R)) < m_{\beta_0}(\mathbf{q}, \mathbf{Q})$ , the triangle inequality gives

$$d(b, \beta(t_R)) \leq d(b, a) + d(a, \beta(t_R)) \leq 1 + m_{\beta_0}(\mathbf{q}, \mathbf{Q}),$$

which we can rewrite as

$$\beta_r \subset \mathcal{N}_1(b, m_{\beta_0}(\mathbf{q}, \mathbf{Q})) \subset \mathcal{N}_\kappa(b, m_{\beta_0}(\mathbf{q}, \mathbf{Q}))$$

which proves the claim.  $\square$

Now we are left to show that the map  $id$  is continuous. Let  $\{c_n\}, c \in \partial_v^D X$  with  $c_n \rightarrow c$ . Assume that  $c_n \rightarrow c$  in  $\partial_v^D X$ , we want to show that  $c_n \rightarrow c$  in  $\partial_\kappa^D X$ . Using the above claim, since each neighbourhood of  $c$  in  $\partial_\kappa^D X$  contains an open neighborhood of  $\partial_v^D X$ , the statement is immediate.  $\square$

**Corollary 4.2.** *For a CAT(0) space  $X$ , the natural map  $i : \partial_\star X \hookrightarrow \partial_\kappa X$  is continuous.*

*Proof.* By Lemma 4.1,  $i_D : \partial_v^D X \hookrightarrow \partial_\kappa X$  is continuous for each  $D$ . Since

$$\partial_\star X = \varinjlim \partial_v^D X,$$

by definition  $i : \partial_\star X \hookrightarrow \partial_\kappa X$  is continuous.  $\square$

**Corollary 4.3.** *Let  $X$  be proper CAT(0) hyperbolic space and let  $\kappa$  be a sublinear function. The space  $\partial_\kappa X$  is compact, and the  $\kappa$ -Morse boundary is homeomorphic to the Gromov boundary.*

*Proof.* Since  $X$  is a hyperbolic space, there exists a uniform constant  $D$  such that every geodesic ray is  $D$ -strongly contracting. This implies that the subspace  $\partial_v^D X$  defined above is the entire visual boundary, in other words, we have  $\partial_v^D X = \partial_v X$ . Also, since every geodesic ray is  $D$ -strongly contracting, the subspace  $\partial_\kappa^D X$  defined above is the full  $\kappa$ -boundary as a topological space. That is to say,  $\partial_\kappa^D X = \partial_\kappa X$  as topological spaces. Proposition 4.1 then yields a homeomorphism between the visual boundary of  $X$ ,  $\partial_v X$  and the  $\kappa$ -boundary of  $X$ ,  $\partial_\kappa X$ . Therefore as topological spaces, we have

$$\partial X \cong \partial_\kappa X.$$

Since  $X$  is proper,  $\partial X$  is compact, thus the  $\kappa$ -boundary  $\partial_\kappa X$  must also be compact.  $\square$

**Theorem 4.4.** *Suppose a group  $G$  acts geometrically on a CAT(0) space  $X$  such that  $\partial_\kappa X \neq \emptyset$ , then the following are equivalent:*

- (1) *Every geodesic ray in  $X$  is  $\kappa$ -contracting.*
- (2) *Every geodesic ray in  $X$  is strongly contracting.*
- (3)  *$\partial_\kappa X$  is compact.*
- (4) *The space  $X$  is hyperbolic.*

*Proof.* We start by showing (3) implies (1). The statement is vacuously true if  $\partial_\kappa X$  is empty. If  $\partial_\kappa X$  is non-empty, then by [Zal22], there exists a rank one isometry  $g$ . This yields the existence of a strongly contracting geodesic line  $l_g$  that is an axis for  $g$ . Let  $\mathfrak{o}$  be a point on  $l_g$  and let  $\beta$  be an arbitrary geodesic ray emanating from  $\mathfrak{o}$ . We show now that  $\beta$  is  $\kappa$ -contracting. Since the action of  $G$  on  $X$  is cocompact, there exists a  $C \geq 0$  and a sequence of group elements  $\{g_i\} \subseteq G$  such

that  $d(\beta(i), g_i \cdot \mathfrak{o}) \leq C$  for each  $i \in \mathbb{N}$  (the black dots in Figure 6). Now, consider the sets given by  $g_i l_g$ . Since  $g_i$  acts by isometry, these are bi-infinite geodesic lines passing the points  $g_i \mathfrak{o}$ . Recall  $[\cdot, \cdot]$  denote a geodesic segment between two points. By CAT(0) geometry, the concatenation of two geodesic segments at angle bounded below by  $\pi/2$  forms a (3,0)-quasi-geodesic segment. Lastly, we denote one end of  $g_i l_g$  by  $g_i l_g(\infty)$  and the other end by  $g_i l_g(-\infty)$ .

For each  $i$ , consider the concatenation

$$[\mathfrak{o}, g_i \mathfrak{o}] \cup [g_i \mathfrak{o}, g_i l_g(\infty)] \text{ and } [\mathfrak{o}, g_i \mathfrak{o}] \cup [g_i \mathfrak{o}, l_g(-\infty)].$$

By CAT(0) geometry, one of these two concatenations consists of geodesic segments intersecting at angles bounded below by  $\pi/2$ . Thus one of the two concatenations is a (3,0)-quasi-geodesic ray starting at  $\mathfrak{o}$ . Relabel the sequence of (3,0)-quasi-geodesic rays defined by concatenating  $[\mathfrak{o}, g_i \mathfrak{o}]$  with either  $[g_i \mathfrak{o}, l_g(\infty)]$  or  $[g_i \mathfrak{o}, l_g(-\infty)]$  to form a sequence of (3,0)-quasi-geodesic rays, by  $\gamma_i$ . Since  $\partial_\kappa X$  is compact, up to passing to a subsequence,  $[\gamma_i]$  converges to an element  $\mathfrak{b} \in \partial_\kappa X$ . Let  $b$  be the geodesic representative of  $\mathfrak{b}$ . The convergence implies that for each  $r > 0$ , there exists  $k$  such that if  $i \geq k$ , the sequence  $\gamma_i$  satisfies

$$\gamma_i|_r \subset \mathcal{N}_\kappa(b, \mathfrak{m}_b(3, 0)).$$

Since the subsegment of  $\gamma_i$  given by  $[\mathfrak{o}, g_i \mathfrak{o}]$  is in a  $C$ -neighbourhood of  $\beta$ , we have for each  $r$ ,  $\beta|_r$  is in  $\mathcal{N}_\kappa(b, C + \mathfrak{m}_b(3, 0))$ , and hence

$$\beta \in \mathcal{N}_\kappa(b, C + \mathfrak{m}_b(3, 0)).$$

Lemma 2.10 then implies that  $\beta = b$ . Thus  $[\beta] = [b] = \mathfrak{b} \in \partial_\kappa X$ , which finishes the proof.

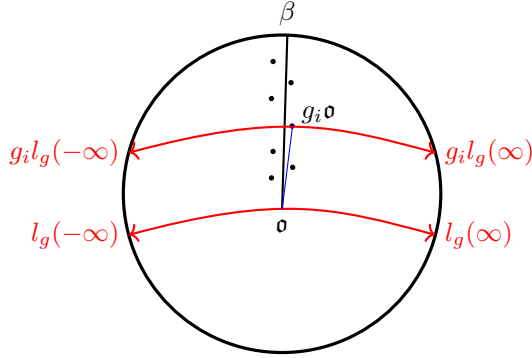


FIGURE 6. Translates of  $l_g$  by  $\{g_i\}$  along  $\alpha_0$

Next we show that (1) implies (4). If every geodesic ray is  $\kappa$ -contracting, then  $X$  does not contain an isometric copy of  $\mathbb{E}^2$ , and hence, by the Flat Plane Theorem ([BH99] III.Γ.3 Theorem 3.1), the space  $X$  must be hyperbolic. The implication (4)  $\Rightarrow$  (3) is Corollary 4.3.

Lastly, we prove the equivalence between (2) and (4). Since every geodesic ray is  $\mathbb{N}$ -Morse for the same  $\mathbb{N}$  in a  $\delta$ -hyperbolic space, we have (4)  $\Rightarrow$  (2). On the other hand, by way of contradiction, suppose  $X$  is not a hyperbolic space, then it must contain isometrically a copy of  $\mathbb{E}^2$  by the Flat Plane Theorem (Flat Plane Theorem

( [BH99] Theorem 3.1). Let  $\mathfrak{o} \in \mathbb{E}^2$  and the geodesic rays that stays entirely in the is not  $D$ -strongly contracting for any  $D$ . Therefore, (2)  $\Rightarrow$  (4).  $\square$

**4.1. Contracting geodesic rays and quasi-möbius maps.** We remind the reader of a few terminologies from [CM17] and [CCM19]. Recall that a geodesic  $\gamma$  is *strongly contracting* if it is in the sublinearly Morse boundary whose associated sublinear function  $\kappa = 1$ . This implies the existence of a constant  $D$  such that all disjoint balls project onto  $\gamma$  to a set of diameter at most  $D$ , in which case we say  $\gamma$  is  *$D$ -strongly contracting*. Consider the set of all  $D$ -strongly-contracting geodesic rays emanating from  $\mathfrak{o}$ . We can think of this set as a subspace of the various boundaries we study in this paper: we use  $\partial_v^D X$  to denote the set of all  $D$ -contracting geodesic rays emanating from  $\mathfrak{o}$  when equipped with the subspace topology of the visual boundary, and use  $\partial_\kappa^D X$  when equipped with the subspace topology of the  $\kappa$ -boundary.

Denote by  $\partial_\kappa^{(n,D)} X$  the collection of all  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  of distinct points  $a_i \in \partial_\kappa X$  such that every bi-infinite geodesic connecting  $a_i$  to  $a_j$  is  $D$ -strongly contracting.

**Definition 4.5.** Let  $X, Y$  be proper geodesic CAT(0) space.

- A map  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  is said to be *1-stable* if for every  $D$ , there exists  $D'$  such that  $f(\partial_\kappa^D X) \subseteq \partial_\kappa^{D'} Y$ .
- A map  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  is said to be *2-stable* if for every  $D$ , there exists  $D'$  such that

$$f(\partial_\kappa^{(2,D)} X) \subseteq \partial_\kappa^{(2,D')} Y.$$

Notice that it follows from the above definition that a 2-stable map  $f$  maps  $\partial_\kappa^{(n,D)} X$  to  $\partial_\kappa^{(n,D')} X$  for all  $n \geq 2$ . Hence, it makes sense to make the following definition.

A map  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  is said to be *stable* if it is both 1 and 2 stable.

**Definition 4.6.** The cross-ratio of a four-tuple  $(a, b, c, d) \in \partial_\kappa^{(4,D)} X$  is defined to be  $[a, b, c, d] = \pm \sup_{\alpha \in (a,c)} d(\pi_\alpha(b), \pi_\alpha(d))$ , where the sign is positive if the orientation

of the geodesic  $(\pi_\alpha(b), \pi_\alpha(d))$  agrees with that of  $(a, c)$  and is negative otherwise.

A stable map  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  is said to be *successively quasi-möbius* if for every  $D$ , there exists a continuous map  $\psi_D : [0, \infty) \rightarrow [0, \infty)$  such that for all 4-tuples  $(a, b, c, d) \in \partial_\kappa^{(4,D)} X$ , we have  $[f(a), f(b), f(c), f(d)] \leq \psi_D([a, b, c, d])$ .

Let  $X_1 \subset X_2 \subset X_3 \subset \dots$  be a nested sequence of topological spaces. The *direct limit* of  $\{X_i\}$ , denoted by  $\varinjlim X_i$ , is the space consisting of the union of all  $X_i$  given the following topology: A subset  $U$  is open in  $\varinjlim X_i$  if  $U \cap X_i$  is open in  $X_i$  for each  $i$ .

The following is a standard way to establish continuous maps between two nested sequences and the proof is left as an exercise for interested readers.

**Lemma 4.7.** *Let  $\{X_i\}$ , and  $\{Y_j\}$  be two sequences of nested topological spaces. Let*

$$X = \varinjlim X_i \text{ and } Y = \varinjlim Y_i$$

*be the direct limit of  $\{X_i\}$  and  $\{Y_i\}$  respectively. If  $f : X \rightarrow Y$  is a map such that:*

- *For each  $i$  there exists some  $j$  with  $f(X_i) \subseteq Y_j$ .*
- *$f|_{X_i} : X_i \rightarrow Y_j$  is continuous.*

Then  $f$  is continuous.

Consider the topological spaces  $\partial_v^D X$ . The Morse boundary  $\partial_* X$  is the direct limit of the topological spaces  $\partial_v^D X$  where  $D \in \mathbb{N}$ . In other words

$$\partial_* X = \varinjlim \partial_v^D X$$

Hence, a set  $U$  is open in  $\partial_* X$  if and only if  $U \cap \partial_v^D X$  is open for each  $D$ . We will often make use of the following theorem.

**Theorem 4.8** ([CCM19]). *Let  $X, Y$  be proper cocompact  $CAT(0)$  spaces with at least 3 points in their Morse boundaries. A homeomorphism  $f : \partial_* X \rightarrow \partial_* Y$  is induced by a quasi-isometry  $h : X \rightarrow Y$  if and only if  $f$  is 2-stable and successively quasi-möbius.*

### 5. SUCCESSIVELY QUASI-MÖBIUS HOMEOMORPHISMS ON THE $\kappa$ -BOUNDARIES

In [Pau96], the author characterizes homeomorphisms between boundaries of cocompact hyperbolic spaces that are induced by quasi-isometries. They characterize such homeomorphisms as the ones that are *quasi-möbius*. In this section, as an application of visibility [Zal22] and using work of [CM17], [CCM19], we prove a weaker version of this characterization:

**Theorem 5.1.** *Let  $X, Y$  be proper cocompact  $CAT(0)$  spaces with at least 3 points in their sublinear boundaries. A homeomorphism  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  is induced by a quasi-isometry  $h : X \rightarrow Y$  if and only if  $f$  is stable and successively quasi-möbius.*

The following is an immediate consequence.

**Corollary 5.2.** *Let  $G$  and  $H$  be  $CAT(0)$  groups. Then  $G$  is quasi-isometric to  $H$  if and only if there exists a homeomorphism  $f : \partial_\kappa G \rightarrow \partial_\kappa H$  which is successively quasi-möbius and stable.*

**Lemma 5.3.** *A quasi-isometry  $h : X \rightarrow Y$  induces a stable homeomorphism  $\partial_\kappa h : \partial_\kappa X \rightarrow \partial_\kappa Y$ .*

*Proof.* Fix  $\mathfrak{o} \in X$  and let  $\mathfrak{o}' = h(\mathfrak{o})$  where  $h$  is a  $(k, K)$ -quasi-isometry. Qing and Rafi show that a quasi-isometry  $h$  induces a homeomorphism  $\partial h$  on their respective  $\kappa$ -boundaries. If  $\gamma$  is a  $D$ -strongly contracting geodesic ray, then by [CS15], the unique geodesic ray starting at  $h(\mathfrak{o})$  and representing  $[f(\gamma)]$  must be  $D'$ -strongly contracting where  $D'$  depends on  $D, k$  and  $K$ . This implies that  $\partial_\kappa h$  is 1-stable. Now, Theorem 4.8 gives us that the map induced by  $h$  on the Morse boundary is 2-stable. Hence, we deduce that  $\partial_\kappa h$  is stable.  $\square$

**Lemma 5.4.** *Any homeomorphism  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  such that  $f, f^{-1}$  are 1-stable induces a homeomorphism  $g : \partial_* X \rightarrow \partial_* Y$  on their Morse boundaries, with  $g(x) = f(x)$  for all  $x \in \partial_* X$ .*

*Proof.* Let  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  be a homeomorphism such that  $f$  and  $f^{-1}$  are 1-stable. Notice that by Theorem E in [QRT22a], we have that if  $\kappa' < \kappa$ , then the inclusion map

$$i : \partial_{\kappa'} X \rightarrow \partial_\kappa X$$

is continuous. Taking  $\kappa' = 1$ , yields that  $i : \partial_1 X \rightarrow \partial_\kappa X$  is continuous. Hence, since both  $f$  and  $f^{-1}$  are 1-stable, the restriction of  $f$  to  $\partial_1 X$  induces a homeomorphism



$\bar{f} : \partial_1 X \rightarrow \partial_1 Y$ , with  $\bar{f} = f|_{\partial_1 X}$  where  $\partial_1 X$  and  $\partial_1 Y$  are given the subspace topology of the  $\kappa$ -boundary. Meanwhile,

$$\partial_1 X = \bigcup_{D=1}^{\infty} \partial_1^D X \text{ and } \partial_1 Y = \bigcup_{D=1}^{\infty} \partial_1^D Y.$$

Since  $\partial_1^D X$  is equipped with the subspace topology of  $\partial_1 X$ , the inclusion map

$$i^D : \partial_1^D X \hookrightarrow \partial_1 X$$

is continuous. Using Lemma 4.1, we get that

$$i^D : \partial_v^D X \hookrightarrow \partial_1 X$$

is continuous for every  $D$ , where  $\partial_v^D X$  is given the subspace topology of the visual boundary. Furthermore, since  $f$  is 1-stable, we have  $\bar{f} \circ i^D : \partial_v^D X \hookrightarrow \partial_1^D Y$  for some  $D'$  where  $\bar{f} \circ i^D$  is continuous. Using Lemma 4.1, we obtain a continuous map

$$\bar{f} \circ i^D : \partial_v^D X \hookrightarrow \partial_v^{D'} Y$$

for each  $D$ . Hence, by Lemma 4.7, we get a continuous map  $g : \partial_* X \rightarrow \partial_* Y$ . Applying the same argument above to  $f^{-1}$  yields a continuous map  $g' : \partial_* Y \rightarrow \partial_* X$  with

$$g \circ g' = id_{\partial_* X} \text{ and } g' \circ g = id_{\partial_* Y},$$

which finishes the proof. □

### 5.1. Proof of Theorem 5.1.

*Proof.* ( $\Rightarrow$ ) If  $h$  is a quasi-isometry, then  $f := \partial h$  is stable by Lemma 5.3. Also,  $f$  is successively quasi-möbius by Theorem 4.8.

( $\Leftarrow$ ) Using Lemma 5.4 any stable homeomorphism  $f : \partial_\kappa X \rightarrow \partial_\kappa Y$  induces a homeomorphism  $g : \partial_* X \rightarrow \partial_* Y$  on their Morse boundaries, with  $g(x) = f(x)$  for all  $x \in \partial_* X$ . Since  $f$  is successively quasi-möbius and  $g(x) = f(x)$  for  $x \in \partial_* X$ , Theorem 4.8 implies the existence of a quasi-isometry  $h : X \rightarrow Y$  such that  $\partial h = g : \partial_* X \rightarrow \partial_* Y$ . We wish to show that the induced map

$$\partial_\kappa h : \partial_\kappa X \rightarrow \partial_\kappa Y$$

agrees with  $f$ . Notice that as a set  $\partial_* X = \partial_1 X$ , where  $\partial_1 X$  is the subset of  $\partial_\kappa X$  consisting of equivalence classes having a strongly contracting representative. Hence, we have  $\partial_\kappa h(x) = \partial h(x)$  for all  $x \in \partial_1 X \subseteq \partial_\kappa X$ . Now, since  $\partial h = g$ , and  $g(x) = f(x)$  on  $\partial_1 X$ , we get that

$$\partial_\kappa h(x) = \partial h(x) = g(x) = f(x)$$

for all  $x \in \partial_1 X$ . Therefore,  $\partial_\kappa h(x) = f(x)$  for all  $x \in \partial_1 X \subseteq \partial_\kappa X$ . It remains to show that  $\partial_\kappa h(x') = f(x')$  for all  $x' \in \partial_\kappa X$ . Let  $x' \in \partial_\kappa X$ , by Corollary 3.7, there exists a sequence  $x_n \in \partial_1 X$  that converges to  $x'$

$$x_n \rightarrow x'$$

in  $\partial_\kappa X$ . Since  $f$  is continuous on  $\partial X_\kappa$ , we have convergence

$$f(x_n) = \partial_\kappa h(x_n) \rightarrow f(x').$$

Also, since  $\partial_\kappa h$  is continuous on  $\partial_\kappa X$ , we get that

$$\partial_\kappa h(x_n) \rightarrow \partial_\kappa h(x').$$

As  $\partial_\kappa Y$  is Hausdorff, we obtain  $\partial_\kappa h(x') = f(x')$ .  $\square$

This result is far from satisfying, since successively quasi-möbius requires one to check the quasi-möbius condition for every  $D$ , it is a much stronger condition than quasi-möbius. Currently there is no results directly characterizing quasi-möbius maps on the  $\kappa$ -boundaries.

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