

DILATATION OF OUTER AUTOMORPHISMS OF RIGHT-ANGLED ARTIN GROUPS

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ABSTRACT. Given a right-angled Artin group defined by a simplicial graph: $A(\Gamma) = \langle V|E \rangle$ and an automorphism $\phi \in \text{Aut}(A(\Gamma))$ there is a natural measure of how fast the length of a word w of $A(\Gamma)$ grows after n iterations of ϕ as a function of n , which we call the dilatation of w under ϕ . We define the dilatation of ϕ as the supremum over dilatations of all $w \in A(\Gamma)$. Assuming that ϕ is a pure and square map, we show that if the dilatation of ϕ is positive, then either there exists a free abelian special subgroup on which the dilatation is realized; or there exists a strata of either free or free abelian groups on which the dilatation is realized.

1. INTRODUCTION

Let $\Gamma = (V, E)$ be a finite simplicial graph. The right-angled Artin group (RAAG) associated to Γ is the group with presentation

$$A(\Gamma) = \langle v \in V \mid [v, w], (v, w) \in E \rangle$$

That is, the generators are in bijection with V , and the only relations are that two generators commute if the corresponding vertices form an edge in Γ . From their simple presentations, we see that RAAGs encompass a spectrum of groups with free groups F_n at one end and free abelian groups \mathbb{Z}^n at the other. RAAGs are a well-studied class of groups with close connections to low-dimensional topology; they arise naturally in the study of diffeomorphism groups of manifolds [KK17] and most recently, they played a key role in Agol's solution [Ago13] to the virtually Haken conjecture for hyperbolic 3-manifolds.

In this article, we study the automorphism groups of RAAGs. At one end of the spectrum, the study of $\text{GL}_n(\mathbb{Z})$ is classical, because of its relation to classification of lattices in \mathbb{R}^n and in turn, flat metrics on tori. At the other, $\text{Out}(\mathbb{F}_n)$, or the outer automorphism group of the free group, became an active area of research beginning in the early 20th century with the work of Nielsen and Magnus [LS77]. The introduction of Culler–Vogtmann Outer space [CV86] provided a geometric action for $\text{Out}(\mathbb{F}_n)$, putting its study on equal footing with that of lattices in semisimple Lie groups and perhaps more closely with the mapping class group of an orientable surface of genus g .

Let $\text{Out}(A(\Gamma))$ denote the outer automorphism group of $A(\Gamma)$. $\text{Out}(A(\Gamma))$ has been shown to possess many of the same properties as $\text{Out}(\mathbb{F}_n)$ and $\text{GL}_n(\mathbb{Z})$. For example, it

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is known to be finitely generated [Ser89],[Lau95], finitely presented [Day09], and to have finite virtual cohomological dimension [CV09], [BCV09], [DW17].

In what follows, we consider the algebraic and dynamical structure of automorphisms of RAAGs, focusing specifically on the asymptotic growth of conjugacy classes of elements under the action of an automorphism. Growth of automorphisms of $GL_n(\mathbb{Z})$ can be easily obtained from the Jordan normal form of a matrix. For $\text{Out}(\mathbb{F}_n)$, on the other hand, a characterization of the growth is obtained by demonstrating the existence of a train track map or a relative train track map, first constructed in [BH92], [BFH00], [BFH05]. In either the free or free abelian case, words grow either polynomially or exponentially under the iterated application of an automorphism.

By work of Day [Day09], any automorphism of $A(\Gamma)$ decomposes uniquely into an \mathbb{F}_n -part and a \mathbb{Z}^n -part, and one may hope that the dynamics of any automorphism admits a similar structural decomposition. However, the construction of train tracks in the free group case relies heavily on Culler–Vogtmann outer space. For certain classes of RAAGs, an outer space has been constructed [CCV07], and Charney–Stambaugh–Vogtmann [CSV17] have constructed an outer space for the “ \mathbb{F}_n ”-part of $\text{Out}(A(\Gamma))$, but no outer space yet exists in general. To overcome this difficulty, we restrict our attention to the collection $\Phi \in \text{Out}(A(\Gamma))$ which map 2-cells to 2-cells without folding the boundary edges (see Definition 2.11), and call such ϕ square maps.

Recall that an element $\Phi \in \text{Out}(A(\Gamma))$ is an equivalence class of automorphisms $\Phi = [\phi]$, defined up to conjugation by elements of $A(\Gamma)$. Given an automorphism $\phi \in \text{Aut}(A(\Gamma))$ and a word $w \in A(\Gamma)$, we define the dilatation $\lambda_\phi(w)$ to be the logarithm of the length of $\phi^n(w)$ divided by n . The dilatation captures the average exponential growth of the length of w under iterated application of ϕ . The *dilatation* λ_ϕ is the supremum of $\lambda_\phi(w)$ over all words $w \in A(\Gamma)$.

Theorem A. *Let $\Phi = [\phi] \in \text{Out}(A(\Gamma))$ be a pure square map. If the dilatation λ_ϕ is positive, then there exists a ϕ -invariant induced subgraph $\Delta \subset \Gamma$ where $\lambda_{\phi|_\Delta} = \lambda_\phi$ and Δ satisfies one of the following:*

- (i) Δ is a complete graph.
- (ii) Δ contains an empty graph Δ_e that is the union of all cycles in \mathcal{D}_Δ .

To prove this theorem we define a finite directed graph \mathcal{D}_Δ which we call the automorphism diagram (cf. §3). The automorphism diagram which keeps track of the verbal substitution of each generator of $A(\Gamma)$ under iterated application of ϕ .

Theorem A can be interpreted as saying that the growth of square automorphisms is not more complicated than that of its free and free abelian subgroups. If the dilatation of a word in $A(\Gamma)$ is zero, this means that its length grows subexponentially. It follows from relative train track theory for free groups [BH92],[BFH00] and from the Jordan normal form for free abelian groups that if an automorphism has vanishing dilatation, then it grows polynomially. We conjecture that the same is true for general RAAG automorphisms, and in Theorem 4.1, we prove that if a pure, square map ϕ has an automorphism diagram without cycles, then $\lambda_\phi = 0$ and ϕ grows polynomially.

The paper is organized as follows. In §2, we review relevant background information on centralizers in right-angled Artin groups and automorphisms, and define the notions of dilatation of an automorphism and of a square map for right-angled Artin groups. In §3, we introduce the main technical tool of the paper, the automorphism diagram, and analyze its key properties for square maps. We show that an automorphism with acyclic automorphism diagram has polynomial growth in §4. Finally in §5, we apply the results of §4 to find the system of free and free abelian subgraphs in Theorem A.

2. BACKGROUND

2.1. Finite simple graphs. Let $\Gamma = (V, E)$ be a *finite simple graph*. That is, a graph in which each edge is uniquely determined by a pair of distinct vertices. By an *induced subgraph* we mean a subgraph $\Gamma' = (V', E')$ such that $V' \subseteq V$ and given $v_1, v_2 \in V'$

$$(v_1, v_2) \in E' \iff (v_1, v_2) \in E.$$

Given a vertex $v_0 \in V$, the *link* of v_0 is the induced subgraph on the set of vertices

$$\{v \mid (v, v_0) \in E\}.$$

In contrast, the *star* of a vertex $v_0 \in V$ is the induced subgraph on

$$\{v \mid (v, v_0) \in E\} \cup \{v_0\}.$$

The *complement* of $\Gamma = (V, E)$, denoted $\bar{\Gamma}$, is the graph (V, E') where

$$(v_1, v_2) \in E' \iff (v_1, v_2) \notin E.$$

2.1.1. Directed graphs. By a *directed graph* we mean a finite graph where each edge is an ordered pair (v_i, v_j) and $(v_i, v_j) \neq (v_j, v_i)$. We also write $v_i \rightarrow v_j$ for a directed edge from v_i to v_j . We can depict a directed graph by putting an arrow on each edge. A path γ is a sequence of vertices

$$v_0 \rightarrow v_1 \rightarrow v_2 \dots \rightarrow v_n$$

A *cycle* is a path as above such that $v_0 = v_n$. We will refer to a path by the tuple of vertices *i.e.* $\gamma = (v_0, \dots, v_n)$.

Remark 2.1. In contrast to standard definition of cycle in graph theory, we allow repeated vertices.

2.2. Right-angled Artin groups. Let $A(\Gamma) = \langle V \mid E \rangle$ denote the *right-angled Artin group* associated to Γ . The generators of $A(\Gamma)$ are in one-to-one correspondence with the vertices V , and two generators commute if and only if their corresponding vertices are connected by an edge in Γ . A *special subgroup* of $A(\Gamma)$ is any subgroup of the form $A(\Delta)$, where $\Delta \subseteq \Gamma$ is an induced subgraph.

Definition 2.2. We fix the generating set once and for all. Given word $w \in A(\Gamma)$, we say w is *reduced* if it cannot be shortened by successively applying commutation relations and canceling vv^{-1} pairs, $v \in V$. We say w is *cyclically reduced* if w and all of its cyclic conjugates are reduced. If w is a reduced, then the *length* of w , denoted $|w|$, is the number

of letters in the spelling of w , and the *support* of w , denoted $\text{supp}(w)$, is the set of elements v in V such that v or v^{-1} occurs in the spelling of w .

2.3. Commuting elements in $A(\Gamma)$. $A(\Gamma)$ is a special case of a graph product, where all the vertex groups are infinite cyclic. The general construction proceeds as follows. Let $\Gamma = (V, E)$ be a graph. To each $v \in V$, associate a vertex group G_v . Then, the *graph product* of the $\{G_v\}$ with respect to the graph (V, E) is defined as F/R where F is the free product of the G_v and R is the normal subgroup generated by subgroups of the form $[G_u, G_v]$ whenever there is an edge joining u and v .

In [Ser89], Servatius studied graph products of groups and characterized when the support of two words commute pairwise. Given two words w_1, w_2 , we will say that $\text{supp}(w_1)$ *commutes with* $\text{supp}(w_2)$ if the elements of $\text{supp}(w_1)$ and $\text{supp}(w_2)$ pairwise commute. We recall the relevant definitions and results here:

Definition 2.3. If $u, v \in A(\Gamma)$ are reduced, then

$$|uv| \leq |u| + |v|.$$

and if equality holds we say that the product uv is a *reduced factorization*. If uv is not reduced, then there exists a unique h such that we have that

$$u = u'h \text{ with } |u| = |u'| + |h| \text{ and } v = h^{-1}v' \text{ with } |v| = |v'| + |h|$$

and

$$uv = u'v' \text{ with } |uv| = |u'| + |v'|.$$

Proposition 2.4. *Suppose that u and v are commuting elements in $A(\Gamma)$ such that*

- (1) *uv is a reduced factorization, and*
- (2) *the induced subgraph of Γ on $\text{supp}(uv)$ has connected complement.*

Then there exists an element h in $A(\Gamma)$ such that both u and v belong to the cyclic subgroup of $A(\Gamma)$ generated by h .

Lemma 2.5. *If a graph $\Gamma = (V, E)$ has its complement $\bar{\Gamma}$ disconnected, then the graph is a join. That is to say, there exists a partition of vertices into two set V_1, V_2 , such that for all $v_1 \in V_1$ and $v_2 \in V_2$,*

$$(v_1, v_2) \in E$$

Proof. Since $\bar{\Gamma}$ is disconnected, the edge set of $\bar{\Gamma}$, we call \bar{E} can be partitioned into two sets E_1 and E_2 such that all edges in E_i connect vertices in V_i . That is to say, all the edges connecting vertices from V_1 to V_2 is in the complement of $\bar{\Gamma}$, i.e. for every $v_1 \in V_1$ and $v_2 \in V_2$,

$$(v_1, v_2) \in \bar{\bar{\Gamma}} = \Gamma.$$

That is to say there is a partition of vertices into two set V_1, V_2 , such that for all $v_1 \in V_1$ and $v_2 \in V_2$,

$$(v_1, v_2) \in E.$$

□

2.4. **Out(RAAG).** Let $\text{Aut}(A(\Gamma))$ be the automorphism group of $A(\Gamma)$. $\text{Aut}(A(\Gamma))$ is finitely generated [Ser89],[Lau95] by the Laurence–Servatius generators, which fall into the following four types:

- Inversions: $\iota_v : v \mapsto v^{-1}$ and is the identity on all other generators.
- Graph isomorphisms: any automorphism of Γ induces a bijection of V which preserves commutation relations,
- Transvections: if $lk(v) \subseteq st(w)$,

$$\tau_{v,w} : v \mapsto wv$$

and fixes all other generators. $\tau_{v,w}$ is called a *twist* if v and w and a *fold* otherwise.

- Partial conjugations: if C is a component of $\Gamma \setminus st(w)$,

$$\chi_{C,w} : v \mapsto wvw^{-1}$$

and fixes all other generators.

We let $\text{Out}(A(\Gamma))$ denote the outer automorphism group of $A(\Gamma)$; this is the quotient of $\text{Aut}(A(\Gamma))$ by the normal subgroup of automorphisms generated by the action of $A(\Gamma)$ on itself by conjugation. $\text{Out}(A(\Gamma))$ is generated by images of the above Laurence–Servatius generators in the quotient.

Remark 2.6. In what follows we will often define properties of elements $\Phi \in \text{Out}(A(\Gamma))$. Since $\Phi = [\phi]$ is strictly speaking an equivalence class, when we say Φ has some property \mathcal{P} , we will mean that there is some representative $\phi \in \text{Aut}(A(\Gamma))$ which has property \mathcal{P} . As an abuse of notation, we will not distinguish between ϕ and $[\phi]$, unless it is not clear from the context.

2.4.1. *Pure automorphisms of $A(\Gamma)$.* In this section we introduce the notion of a *pure* automorphism in $\text{Out}(A(\Gamma))$. An element $\phi \in \text{Aut}(A(\Gamma))$ is *pure* if it is cyclically reduced and

$$s \in \text{supp}(\phi(s)) \text{ for all } s \in V.$$

An equivalence class $[\phi] \in \text{Out}(A(\Gamma))$ is pure if it has a pure representative in $\text{Aut}(A(\Gamma))$.

Remark 2.7. If ϕ is a pure automorphism, then it follows from definition same representative ϕ^k is also a pure automorphism.

Proposition 2.8. *For every $\phi \in \text{Aut}(A(\Gamma))$, there exists a positive integer N such that ϕ^N is a pure automorphism.*

Proof. The abelianization of $A(\Gamma)$ is obtained from $A(\Gamma)$ by making all generators commute. Hence $A(\Gamma)^{ab} \cong \mathbb{Z}^{|V|}$, generated by the equivalence classes $\{[v] | v \in V\}$. For any word $w \in A(\Gamma)$ we write

$$[w] = \sum_{v \in V} n_v [v] \in \mathbb{Z}^{|V|}$$

where n_v is just the exponent sum of all v 's occurring in w . By reducing coefficients modulo 2, there is a natural surjection $\mathbb{Z}^{|V|} \rightarrow (\mathbb{Z}/2\mathbb{Z})^{|V|}$. Observe that if $n_v \not\equiv 0 \pmod{2}$, then we must have $v \in \text{supp}(w)$.

Since the kernel of the surjection $A(\Gamma) \rightarrow (\mathbb{Z}/2\mathbb{Z})^{|V|}$ is characteristic, any automorphism ϕ of $A(\Gamma)$ induces an automorphism $A_\phi \in \text{GL}(|V|, \mathbb{Z}/2\mathbb{Z})$. The latter has finite order, hence for some N , we have $A_\phi^N = \text{Id}$. But this means that for any $s \in V$, if we write

$$[\phi^N(s)] = \sum_{v \in V} n_v[v],$$

then

$$n_v \equiv \begin{cases} 0 \pmod{2}, & v \neq s \\ 1 \pmod{2}, & v = s \end{cases}$$

Hence ϕ^N is pure, as desired. \square

2.5. Dilatation. Consider an infinite order automorphism ϕ of $A(\Gamma)$. We will be interested in the dynamics of ϕ acting on $A(\Gamma)$. One way to do this is to measure the asymptotic growth of the word length of elements of $A(\Gamma)$ under powers of ϕ .

Definition 2.9. The *dilatation of ϕ at w* is

$$\lambda_\phi(w) := \lim_{k \rightarrow \infty} \frac{\log |\phi^k(w)|}{k},$$

and the *dilatation of ϕ* is

$$\lambda_\phi := \sup_{w \in A(\Gamma)} \lambda_\phi(w)$$

Note that $\lambda_\phi(w)$ always exists since the sequence $\left\{ \frac{\log |\phi^k(w)|}{k} \right\}$ is subadditive. We claim that the dilatation λ_ϕ is always realized by some word w , and in fact, we can take w to be a generator:

Lemma 2.10.

$$\lambda_\phi = \max_{s \in V} \{\lambda_\phi(s)\}$$

In particular, λ_ϕ is always finite and realized by some (not necessarily unique) generator s_i .

Proof. Given any word w in $A(\Gamma)$ we can write $w = s_1 s_2 \cdots s_n$ for some generators $s_i \in V$. Then

$$\begin{aligned} |\phi^k(w)| &= |\phi^k(s_1) \cdots \phi^k(s_n)| \\ &\leq |\phi^k(s_1)| + \cdots + |\phi^k(s_n)| \\ &\leq n \max_i \{|\phi^k(s_i)|\} \end{aligned}$$

Taking log of both sides and dividing by k :

$$\frac{\log |\phi^k(w)|}{k} \leq \frac{\log(n)}{k} + \max_i \left\{ \frac{\log |\phi^k(s_i)|}{k} \right\}$$

Since n is fixed, the first term on the right goes to 0 as $k \rightarrow \infty$. We conclude that for any $w \in A(\Gamma)$

$$\lambda_\phi(w) \leq \max_{s \in V} \{\lambda_\phi(s)\}$$

and therefore

$$\lambda_\phi = \max_{s \in V} \{\lambda_\phi(s)\}.$$

In particular, λ_ϕ exists and is realized by a generator. □

Given $g \in A(\Gamma)$, let ϕ^g denote the composition of ϕ with conjugation by g . It is not difficult to see that $\lambda_\phi = \lambda_{\phi^g}$. Indeed, given $w \in A(\Gamma)$ we have:

$$\begin{aligned} |(g\phi g^{-1})^k(w)| &= |g\phi(g) \cdots \phi^{k-1}(g)\phi^k(w)\phi^{k-1}(g^{-1}) \cdots \phi(g^{-1})g^{-1}| \\ &\leq 2 \left(|g| + |\phi(g)| + \cdots + |\phi^{k-1}(g)| \right) + |\phi^k(w)| \\ &\leq 2k \max\{|g|, |\phi(g)|, \dots, |\phi^{k-1}(g)|, |\phi^k(w)|\} \end{aligned}$$

Now applying log to both sides and dividing by k , this is at most λ_ϕ in the limit, since $\log(2k)/k \rightarrow 0$. Thus, for all w , $\lambda_{\phi^g}(w) \leq \lambda_\phi$. By symmetry, $\lambda_{\phi^g} \geq \lambda_\phi$ and thus $\lambda_\phi = \lambda_{\phi^g}$. Therefore, the dilatation is a well-defined invariant of the equivalence class $[\phi] \in \text{Out}(A(\Gamma))$. If $\lambda_\phi(s) > 0$ or $\lambda_\phi > 0$ is positive, we also say that the element s *grows exponentially* under the map ϕ , or that ϕ is an *exponentially growing map*.

Furthermore, we say a subgraph $\Delta \subseteq \Gamma$, is *invariant* under ϕ if ϕ takes words whose support is in Δ to words whose support is in Δ . Given an invariant subgraph $\Delta \subset \Gamma$, one can define

$$\lambda|_\Delta := \sup_{w \in A(\Delta)} \lambda_\phi(w).$$

2.6. Square maps. If we identify $A(\Gamma)$ with the fundamental group of a Salvetti complex $S(\Gamma)$, then elements of $\text{Out}(A(\Gamma))$ as homotopy classes of maps from $S(\Gamma)$ to itself. Pairs of commuting generators correspond to 2-tori in $S(\Gamma)$, and we will be interested in when $\Phi \in \text{Out}(A(\Gamma))$ folds these tori nicely onto other tori in $S(\Gamma)$. This idea is encoded in the following definition.

Definition 2.11. Fixing the presentation of $A(\Gamma)$ to be the one associated with Γ once and for all. The map $\phi \in \text{Aut}(A(\Gamma))$ is called a *square map* if for all $s_1, s_2 \in S$:

$$s_1 \text{ commutes with } s_2 \quad \implies \quad \text{supp}(\phi(s_1)) \text{ commutes with } \text{supp}(\phi(s_2)).$$

We say that $\Phi \in \text{Out}(A(\Gamma))$ is a square map if there exists a representative ϕ of Φ which is a square map.

We say that an automorphism ϕ is a *positive* automorphism if all generators are mapped to words that do not contain negative powers of any generator.

Lemma 2.12. *A positive automorphism ϕ is a square automorphism.*

Proof. Since $\phi(s_1)$ and $\phi(s_2)$ are primitive elements, they are not powers of any other word. By assumption $\phi(s_1)$ and $\phi(s_2)$ commute. Also, since ϕ is a positive map, both $\phi(s_1)\phi(s_2)$ and $\phi(s_2)\phi(s_1)$ are factorially reduced as defined in Definition 2.3. Therefore, by Proposition 2.4, the subgraph of Γ generated by $\text{supp}(\phi(s_1))$ has disconnected complement. By Lemma 2.5, that means the subgraph of Γ generated by $\text{supp}(\phi(s_1))$ is a join of $\text{supp}(\phi(s_1))$ and $\text{supp}(\phi(s_2))$, which means elements of $\text{supp}(\phi(s_1))$ and $\text{supp}(\phi(s_2))$ pairwise commute. \square

Remark 2.13. We show that square maps are not limited to positive maps with the following example. Let $A(\Gamma) = \langle a, b, c \mid [a, c], [b, c] \rangle$:

$$\begin{aligned}\phi : a &\rightarrow aba^{-1} \\ b &\rightarrow ba^{-1} \\ c &\rightarrow c\end{aligned}$$

Lemma 2.14. *Let ϕ be a square map. Then ϕ^k is also a square map for any positive integer k .*

Proof. We prove the statement by induction on k .

$$\phi^{k-1}(s_1) = s_{11}s_{12}\dots s_{1m} \quad \text{and} \quad \phi^{k-1}(s_2) = s_{21}s_{22}\dots s_{2n}$$

Given that s_{1i} and s_{2j} commutes for all i, j then under the square map ϕ , $\text{supp}(\phi(s_{1i}))$ and $\text{supp}(\phi(s_{2j}))$ pairwise commute for all i, j . That is to say, $\text{supp}(\phi^k(s_1))$ and $\text{supp}(\phi^k(s_2))$ pairwise commute. \square

For the rest of the paper we assume ϕ is a pure, square automorphism.

3. AUTOMORPHISM DIAGRAM

The main tool of this paper is a directed graph \mathcal{D}_ϕ that is defined on the generating set of the group $A(\Gamma)$. We deduce quantitative and structural information on the dilatation of ϕ from the combinatorics of \mathcal{D}_ϕ .

Definition 3.1. The vertex set of \mathcal{D}_ϕ is V . Two vertices s_i, s_j are connected by a directed edge from s_i to s_j if and only if $s_j \in \text{supp}(s_i)$.

In this paper we are only concerned with *pure* automorphisms; however, we do not add directed edges from s_i to itself for any i . That is to say, only *distinct* vertices s_i, s_j can be connected by a directed edge in \mathcal{D}_ϕ . On the other hand, it is possible to have a directed edge from s_i to s_j and from s_j to s_i .

For the rest of the section, assume that ϕ is a pure square map and $\phi \in \text{Out}(A(\Gamma))$. Given a vertex $s \in \mathcal{D}_\phi$, the *down-set* of s is an induced subgraph whose vertex set is the following union:

$$\{v \mid \text{there exists a directed path from } s \text{ to } v\}.$$

The induced graph on this set of vertices we call *down – set* and denote it by $d(s)$. Note that $s \in d(s)$, as the constant path is directed. By construction,

$$(1) \quad \lambda_s = \lambda_{d(s)}.$$

Lemma 3.2. *Suppose there is a directed path in \mathcal{D}_ϕ of length n :*

$$(s_0, s_1, s_2, \dots, s_n)$$

and suppose that s_0 commutes with s_1 . Then $\{s_0, s_1, s_2, \dots, s_n\}$ form a complete subgraph in Γ .

Proof. First we show that s_i and s_{i+1} commute for all i . Since s_0 and s_1 commute, $\phi(s_0)$ and $\phi(s_1)$ commute. By assumption, ϕ is pure and a square map,

thus all elements of $\text{supp}(\phi(s_1))$ commute with all elements of $\text{supp}(\phi(s_2))$. In particular,

$$s_1 \in \text{supp}(\phi(s_0)) \text{ and } s_2 \in \text{supp}(\phi(s_1)),$$

thus, s_1 commutes with s_2 . Applying this argument to s_i and s_{i+1} for all i , we conclude that s_i commutes with s_{i+1} for all i . We finish the proof by proving the following statement: Suppose (s_0, \dots, s_n) is a directed path in \mathcal{D}_ϕ in which s_i commutes with s_{i+1} for all $0 \leq i \leq n-1$, then $\{s_i\}$ form a complete subgraph.

We prove this statement by induction on n . The case $n=1$ is trivial. Now consider a path (s_0, \dots, s_n) such that s_i and s_{i+1} commute for all i . The subpaths (s_0, \dots, s_{n-1}) and (s_1, \dots, s_n) have length $n-1$ hence by induction, it suffices to show that s_0 and s_n commute. Since s_0 and s_{n-1} commute, by the argument above, every element in $\text{supp}(\phi(s_0))$ commutes with every element in $\text{supp}(\phi(s_{n-1}))$. Since ϕ is pure, $s_0 \in \text{supp}(\phi(s_0))$. Given that $s_n \in \text{supp}(\phi(s_{n-1}))$, ϕ is a square map means s_0 commutes with s_n . Therefore, all pairs of elements in $\{s_0, s_1, s_2, \dots, s_n\}$ commute and $\{s_0, s_1, s_2, \dots, s_n\}$ forms a complete subgraph in Γ . □

Now suppose instead of a directed path, there is a cycle $(s_0, s_1, s_2, \dots, s_n, s_0)$ in \mathcal{D}_ϕ ,

Corollary 3.3. *If an adjacent pair of elements of the cycle commute in $A(\Gamma)$, then*

$$\{s_0, s_1, s_2, \dots, s_n\}$$

form a complete subgraph in Γ .

Proof. Suppose s_i and s_{i+1} commute, then consider the path

$$\{s_i, s_{i+1}, s_{i+2}, \dots, s_n, s_1, s_2, \dots, s_{i-1}\}$$

is a directed path in \mathcal{D}_ϕ . By Lemma 3.2, all elements on this path form a complete subgraph in \mathcal{D}_ϕ . □

Corollary 3.4. *If a non-adjacent pair of elements of the cycle commute in $A(\Gamma)$, then*

$$\{s_0, s_1, s_2, \dots, s_k\}$$

form a complete subgroup in Γ .

Proof. Without loss of generality, suppose s_0 commutes with s_j for some $j \neq 2$. We first claim that ϕ^j inherits the desirable properties of ϕ :

Claim 3.5. Let $\Phi = [\phi] \in \text{Out}(A(\Gamma))$ be a pure, square automorphism. Then $\phi^j, j \in \mathbb{N}$ is also a pure, square automorphism.

The fact that ϕ^j is a square automorphism follows from Remark 2.14. The rest of the claim follows from Remark 2.7. □

From Corollary 3.3 and Corollary 3.4 we conclude that,

Proposition 3.6. *Every cycle $C \subseteq \mathcal{D}_\phi$ forms an induced subgraph in Γ that is either a complete graph or an empty graph.*

Proof. Combine Corollary 3.3 and Corollary 3.4; if there is any pair of elements of C that commutes then elements of C form a complete subgraph in Γ . Otherwise no pair of elements of C commute and the elements of C form an empty graph in Γ . □

Moreover, if two cycles in D_ϕ share a vertex, then the union of all vertices support an empty or a complete subgraph in Γ :

Corollary 3.7. *Let $s \in V$ be part of two (directed) cycles C_1, C_2 , and there is a pair of commuting elements in $C_1 \cup C_2$, then the union of the vertices of C_1 and C_2 form a complete subgraph in Γ .*

Proof. First suppose the two commuting vertices belong to one C_i and without loss of generality that $i = 1$. By Corollary 3.4, elements of C_1 form a complete subgraph. Every element of C_1 is connected by a directed path to every element of C_2 . Indeed, let $s = s_{(1,1)} = s_{(2,1)}$. To connect the i -th element in C_1 , $s_{(1,i)}$ and the j -th element in C_2 , $s_{(2,j)}$, we have:

$$s_{(1,i)}, s_{(1,i+1)}, s_{(1,i+2)} \cdots s_{(1,1)} = s_{(2,1)}, s_{(2,2)}, \dots, s_{(2,j)}.$$

Therefore by Lemma 3.2, every element of C_1 commutes with every element of C_2 . It remains to consider elements of C_2 .

It remains to consider the case when the commuting pair consists of one vertex from each cycle. That is, there exists $s_{(1,i)} \in C_1$ and $s_{(2,j)} \in C_2$ commute. There exists a cycle that contains $s_{(1,i)}$ and $s_{(2,j)}$. Indeed,

$$s_{(1,i)}, s_{(1,i+1)} \cdots s = s_{(2,1)}, s_{(2,2)}, \dots, s_{(2,j)}, s_{(2,j+1)} \cdots s = s_{(1,1)}, s_{(1,2)} \cdots s_{(1,i)}$$

forms a cycle. By Corollary 3.4, all elements in $C_1 \cup C_2$ form a complete subgraph. □

Remark 3.8. In short, it suffices to realize that

$$s = s_{(1,1)}, s_{(1,2)} \cdots s = s_{(2,1)}, s_{(2,2)}, \dots, s$$

is a cycle (with repeated vertex s), applying Corollary 3.3 and Corollary 3.4 proves Corollary 3.7.

Lastly, we use $\overline{\mathcal{D}_\phi}$ to denote the underlying graph of \mathcal{D}_ϕ , that is, the graph whose vertex set is the vertex set of \mathcal{D}_ϕ , and (v, v') is an (undirected) edge in $\overline{\mathcal{D}_\phi}$ if there is a directed edge in \mathcal{D}_ϕ either from v to v' or from v' to v . We say that \mathcal{D}_ϕ is *connected* if $\overline{\mathcal{D}_\phi}$ is connected. Otherwise we say that \mathcal{D}_ϕ is *disconnected*. We make the following observation about the dilatation of ϕ when \mathcal{D}_ϕ is disconnected.

Lemma 3.9. *Suppose the connected components of \mathcal{D}_ϕ is $(\mathcal{D}_\phi)_1, (\mathcal{D}_\phi)_2, \dots, (\mathcal{D}_\phi)_k$, then there exists a $(\mathcal{D}_\phi)_i$ such that*

$$\lambda_\phi = \lambda|_{(\mathcal{D}_\phi)_i}.$$

Proof. For each generator s , by Equation 1,

$$\lambda_s = \lambda|_{d(s)}$$

If $s \in (\mathcal{D}_\phi)_j$, then $d(s) \subset (\mathcal{D}_\phi)_j$, therefore

$$\lambda|_{(\mathcal{D}_\phi)_j} \geq \lambda|_{d(s)} = \lambda_s = \lambda_\phi \geq \lambda|_{(\mathcal{D}_\phi)_j}.$$

Therefore

$$\lambda_\phi = \lambda|_{(\mathcal{D}_\phi)_j}$$

for the component that contains the dilatation-realizing generator. \square

4. GROWTH OF GENERATORS

Given $\Gamma(A)$ and an automorphism ϕ , we say a word w grows *polynomially* under iterations of ϕ if there exists n_0 , a constant C and a polynomial function $p(n)$ such that

$$|\phi^n(w)| \leq Cp(n) \quad \forall n > n_0.$$

If $w = s_0$ is the generator that realizes the dilatation of ϕ , and s_0 grows polynomially under the map ϕ , or then we say that ϕ is an *polynomially growing map*.

Theorem 4.1. *If there is no cycle in \mathcal{D}_ϕ , then ϕ is a polynomially growing map.*

Before proving this theorem, we need some preliminary definitions for the case when \mathcal{D}_ϕ does not have any cycle.

Definition 4.2. A vertex $s \in \mathcal{D}_\phi$ is *terminal* if $d(s) = \emptyset$.

There is a natural partition of the vertices of $s \in \mathcal{D}_\phi$, defined as follows. Let \mathcal{T}_0 be the set of terminal vertices and inductively define \mathcal{T}_i to be the set of terminal vertices of

$$\mathcal{D}_\phi \setminus \cup_{j=0}^i \mathcal{T}_j.$$

Definition 4.3. The collection $\mathcal{T} = \{\mathcal{T}_i\}$ is called the *terminal partition* of V . The *height* of \mathcal{T} is the least k such that $\cup_{j=0}^k \mathcal{T}_j = V$.

We now proceed with the proof of the theorem.

Proof. Suppose \mathcal{D}_ϕ has no cycle. We will in fact show that ϕ has polynomial growth of degree bounded by the height h of \mathcal{T} . More precisely, we will show that there exists a polynomial $p(n)$ of degree at most h such that for any $w \in A(\Gamma)$ there exists a constant C such that

$$|\phi^n(w)| \leq Cp(n).$$

Let $W_i = \cup_{j=0}^i \mathcal{T}_j$ and let G_i be the subgroup generated by W_i , and set $G_{-1} = \{1\}$. Clearly, ϕ preserves G_i . Choosing $s \in \mathcal{T}_i$, we can write

$$\phi(s) = t_0 s^{\epsilon_1} \cdots s^{\epsilon_k} t_k.$$

where $\epsilon_i \in \mathbb{Z}$, and $t_j \in G_{i-1}$.

Claim 4.4. In the above decomposition, $k = 1$, i.e. $\phi(s) = t_0 s^{\pm 1} t_1$.

Proof. The proof is by induction on the height h of \mathcal{T} . For the base case let $A(\Gamma)$ be a right-angled Artin group with height zero, $h = 0$. Then for each $s \in V$, $\phi(s) = s^m$ for all $s \in V$ and some $m \in \mathbb{Z}$. Since ϕ is an automorphism, $m = \pm 1$. Now assume the claim has been established for all RAAGs $A(\Gamma)$ and for any diagram of height at most h without cycles, and suppose \mathcal{D}_ϕ is a diagram of height $h + 1$. In particular, $G_{h+1} = A(\Gamma)$. The key observation is that G_h is a special subgroup and that $\phi|_{G_h} = \phi_h$ is an automorphism of G_h whose corresponding diagram has height h and no cycle. By the induction hypothesis, for $0 \leq i \leq h$ and for any $s \in \mathcal{T}_i$, we have that

$$\phi(s) = t_0 s t_1$$

where $t_j \in G_{i-1}$. We show that there exists an automorphism ψ such that $\tau = \psi^{-1} \circ \phi$ is an automorphism of height at most 1 and $\psi(s) = u_s s u_s^{-1}$ for all $s \in \mathcal{T}_{h+1}$. Indeed, $\phi_h \in \text{Aut}^0(G_h)$ and is therefore a product of partial conjugations and transvections. On the other hand, ϕ_h is in the image of the restriction homomorphism $\rho_h : \text{Aut}^0(G_{h+1}; G_h) \rightarrow \text{Aut}^0(G_h)$. We can lift ϕ_h to $\psi : A(\Gamma) \rightarrow A(\Gamma)$ at the expense of some element in $\ker(\rho_h)$. Observe that transvections lift directly, and partial conjugations lift at the expense of possibly conjugating elements of \mathcal{T}_{h+1} by elements of W_h . Thus ψ satisfies the following two properties:

- (1) $\psi|_{G_h} = \phi_h$,
- (2) for every $s \in \mathcal{T}_{h+1}$, we have

$$\psi(s) = u_s s u_s^{-1},$$

where $u_s \in G_h$.

Now $\tau = \psi^{-1} \circ \phi$ is a pure automorphism whose diagram \mathcal{D}_τ also has no cycle, and which has height at most 1, since

$$\tau|_{G_h} = \psi^{-1}|_{G_h} \circ \phi|_{G_h} = \phi_h^{-1} \circ \phi_h = \text{id}.$$

By induction, we know that for every $s \in \mathcal{T}_{h+1}$, we have $\tau(s) = t'_0 s^{\pm 1} t'_1$, where s is not contained in $\text{supp}(t'_0)$ nor $\text{supp}(t'_1)$. From property (2), applying ψ to $\tau(s)$, we have

$$\phi(s) = \psi \circ \tau(s) = \psi(t'_0 s^{\pm 1} t'_1) = \psi(t'_0)(u_s s^{\pm 1} u_s^{-1})\psi(t'_1).$$

As s does not occur in the support of $\psi(t'_0)$, u_s , or $\psi(t'_1)$, we conclude that $k = 1$ for all $s \in V$, as desired. \square

We now use Claim 4.4 to show that there is a polynomial $p(n)$ such that for any $t \in G_i$ there exists $C > 0$ with

$$|\phi^n(t)| \leq Cp(n),$$

where p is a polynomial of degree at most i . The proof is by induction on i , where the base case is the observation above that $\phi(s) = s$ for $s \in \mathcal{T}_0$. Suppose we have established the bound for all $t \in G_i$, and choose $s \in \mathcal{T}_{i+1}$. By Claim 4.4 we can write

$$\phi(s) = t_0 s t_1,$$

where the t_j , $j = 1, 2$ are words in G_i . Let C_j , $j = 1, 2$ be constants such that

$$|\phi^n(t_j)| = C_j p(n)$$

and set $C = \max\{C_j\}$. Then we have

$$\begin{aligned} |\phi^n(s)| &\leq |\phi^{n-1}(t_0)| + |\phi^{n-1}(s)| + |\phi^{n-1}(t_1)| \\ &\leq 2Cp(n) + |\phi^{n-1}(s)| \end{aligned}$$

By iterating this procedure we have

$$|\phi^n(s)| \leq 2C \sum_{r=1}^{n-1} p(r) + 1.$$

We know that the partial sum of a polynomial of degree d can be written as a polynomial of degree $d+1$. It follows that the right-hand side can be written as a polynomial of degree at most $i+1$. Finally, an arbitrary word in G_{i+1} can be written as a finite product of elements in some \mathcal{T}_j , $0 \leq j \leq i+1$. To finish the proof of the lemma, note that $G_i = A(\Gamma)$ for i large. \square

Corollary 4.5. *Given a generator s and ϕ , if there is not a cycle in $d(s)$, then s grows polynomially.*

Proof. This follows from Equation 1. \square

5. INVARIANT SUBGROUPS

Now we are ready to prove the existence of a complete or empty subgraph inside a subgraph that is invariant under exponentially growing square maps.

Theorem 5.1. *Let $\Phi = [\phi] \in \text{Out}(A(\Gamma))$ be a pure square map. If the dilatation λ_ϕ is positive, then there exists a ϕ -invariant, induced subgraph $\Delta \subset \Gamma$ where $\lambda|_\Delta = \lambda_\phi$, and Δ is of one of the following cases:*

- (i) Δ is a complete graph.
- (ii) Δ contains an empty graph Δ_e that is the union of all cycles in \mathcal{D}_Δ .

Proof. By Lemma 2.10, there exists a generator s that realizes the dilatation of the group, let's denote it s_0 :

$$\lambda_{s_0} = \lambda_\phi.$$

Consider the down-set of s_0 in \mathcal{D}_ϕ , recall $\lambda_{d(s_0)}$ is well defined and $\lambda_{d(s_0)} = \lambda_{s_0} = \lambda_\phi$. By assumption there exists a cycle in $\mathcal{D}_{d(s_0)}$. Now consider a “trim” of $d(s_0)$: a directed edge is *outgoing* with respect to a vertex v if the edge is from v to v' for some other v' . If a vertex in a directed graph has only outgoing edges, then we say the vertex is a *source*. If a vertex has only incoming edges, then we say the vertex is a *sink*. Consider the vertices of $d(s_0)$ and consider the set of all the non-source vertices:

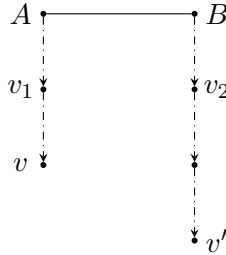
$$[d(s_0)]_1 := \{v | v \in d(s_0) \text{ and } v \text{ is not a source}\}.$$

By an abuse of notation we also use $[d(s_0)]_1$ to denote the induced directed graph on this set. $[d(s_0)]_1$ is ϕ -invariant. Now we trim $[d(s_0)]_1$ likewise: subtract from $[d(s_0)]_1$ any and all of its source vertices and all directed edges incident to the source vertices. The trimming process ends after finite steps since there exists a cycle in $d(s_0)$. The resulting directed graph we denote by $[d(s_0)]_t$, which is also ϕ -invariant and contains a cycle. Since $[d(s_0)]_t$ is ϕ -invariant, one can consider the dilatation of ϕ restricted to $[d(s_0)]_t$. It is noted that the trimming process does not change the dilatation, since a word of arbitrary length visit a source vertex a finite number of times. Thus,

$$\lambda_\phi = \lambda_{s_0} = \lambda_{d(s_0)} = \lambda_{[d(s_0)]_1} = \lambda_{[d(s_0)]_2} = \dots = \lambda_{[d(s_0)]_t}.$$

Suppose $[d(s_0)]_t$ has exactly one component in \mathcal{D}_ϕ . Let $\Delta := [d(s_0)]_t$. Since Δ contains at least one cycle, let Δ_e denote the subgraph of Δ that is the union of all cycles. Δ_e is unique by maximality. If a pair of elements in Δ_e commutes, then by Corollary 3.3 and Corollary 3.4, Δ_e is a complete graph. Furthermore, consider a vertex $v \in \Delta$ and $v \notin \Delta_e$. Since Δ is source-less and connected, there exists a directed path starting from an edge in Δ_e and ending at v . Therefore by Lemma 3.2, v forms a complete graph with Δ_e . Moreover, we show that two such vertices necessarily commute in the following lemma:

Lemma 5.2. *Let ϕ be a pure, square map. Let A, B be a pair of commuting vertices and let v, v' be two vertices that are each connected by a directed path from A and B , respectively. Then v commutes with v' in $A(\Gamma)$.*



Vertices that are in the down-set of two commuting vertices commute.

Proof. Let v_1 be the first vertex on the directed path from A to v and v_2 be the first vertex on the directed path from B to v' . By an abuse of notation, we use $v, v_1, v_2, etc.$ to denote also the generator that they represent in $A(\Gamma)$. Since $v_1 \in \text{supp}(\phi(A))$ and $B \in \text{supp}(\phi(B))$, by Definition 2.11, v_1 and B commute. Likewise v_1 and v_2 commute. Using Definition 2.11, we can show that v_1 commute with all successive elements of the directed path originating from B , therefore v_1 commute with v' . Now consider the next vertex on the direct path from A , call it v_{11} . Using Definition 2.11 again one can show that v_{11} commute with all successive elements of the directed path originating from B and hence v' . By moving down the directed path from A , one show that v and v' commute. \square

Therefore all vertices that are in $\Delta \setminus \Delta_e$ commute to each other, therefore, Δ is a complete subgraph in Γ . On the other hand, if there is no edge in Δ_e then Δ contains an empty graph Δ_e that is the union of all cycles in \mathcal{D}_Δ .

It remains to consider the case when Δ is not connected. In this case, each connected component of $[d(s_0)]_t$ is either a complete graph or contains an empty graph that is a union of all cycles in that component in \mathcal{D}_ϕ . By Lemma 3.9, there exists a component of Δ with desired property that realizes the dilatation. \square

REFERENCES

- [Ago13] Ian Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.
- [BCV09] Kai-Uwe Bux, Ruth Charney, and Karen Vogtmann. Automorphisms of two-dimensional RAAGS and partially symmetric automorphisms of free groups. *Groups Geom. Dyn.*, 3(4):541–554, 2009.
- [BFH00] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\text{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms. *Ann. of Math. (2)*, 151(2):517–623, 2000.
- [BFH05] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\text{Out}(F_n)$. II. A Kolchin type theorem. *Ann. of Math. (2)*, 161(1):1–59, 2005.
- [BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.
- [CCV07] Ruth Charney, John Crisp, and Karen Vogtmann. Automorphisms of 2-dimensional right-angled Artin groups. *Geom. Topol.*, 11:2227–2264, 2007.
- [CSV17] Ruth Charney, Nathaniel Stambaugh, and Karen Vogtmann. Outer space for untwisted automorphisms of right-angled Artin groups. *Geom. Topol.*, 21(2):1131–1178, 2017.
- [CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.
- [CV09] Ruth Charney and Karen Vogtmann. Finiteness properties of automorphism groups of right-angled Artin groups. *Bull. Lond. Math. Soc.*, 41(1):94–102, 2009.
- [Day09] Matthew B. Day. Peak reduction and finite presentations for automorphism groups of right-angled Artin groups. *Geom. Topol.*, 13(2):817–855, 2009.
- [DW17] Matthew B. Day and Richard D. Wade. Relative automorphism groups of right-angled artin groups. Preprint: <https://arxiv.org/abs/1712.01583>, 2017.
- [KK17] Sang-hyun Kim and Thomas Koberda. RAAGs in diffeos. In *Hyperbolic geometry and geometric group theory*, volume 73 of *Adv. Stud. Pure Math.*, pages 215–224. Math. Soc. Japan, Tokyo, 2017.
- [Lau95] Michael R. Laurence. A generating set for the automorphism group of a graph group. *J. London Math. Soc. (2)*, 52(2):318–334, 1995.

- [Lev09] Gilbert Levitt. Counting growth types of automorphisms of free groups. *Geom. Funct. Anal.*, 19(4):1119–1146, 2009.
- [LS77] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin-New York, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.
- [Ser89] Herman Servatius. Automorphisms of graph groups. *J. Algebra*, 126(1):34–60, 1989.

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