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# Subspace intersection graphs

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ABSTRACT

Given a set $R$ of affine subspaces in $\mathbb{R}^d$ of dimension $e$, its intersection graph $G$ has a vertex for each subspace, and two vertices are adjacent in $G$ if and only if their corresponding subspaces intersect. For each pair of positive integers $d$ and $e$ we obtain the class of $(d, e)$-subspace intersection graphs. We classify the classes of $(d, e)$-subspace intersection graphs by containment, for $e = 1$ or $e = d - 1$ or $d \leq 4$.

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## 1. Introduction and preliminary results

Given a family of sets $S_1, S_2, \ldots, S_n$, one can form the intersection graph $G$ of the family by associating a vertex of $G$ with each set and joining two distinct vertices with an edge if their corresponding sets have a nonempty intersection. Conversely, any finite graph can be viewed as the intersection graph of a family of sets in many different ways. If each set is a line segment in the plane, then $G$ is called a *segment intersection graph* [1]. The most interesting problem for segment intersection graphs has been Scheinerman's conjecture, which asks whether all planar graphs are segment intersection graphs [2]. Hartman et al. [5] and deCastro et al. [4] answered this question in the affirmative in the cases of bipartite and triangle-free graphs, respectively. Recently Chalopin and Gonçalves have announced a proof that all planar graphs are segment intersection graphs [3].

If each set $S_i$ is an affine subspace of $k^d$ for some field $k$, then $G$ has an *affine representation* in $k^d$. For a given $k$, the smallest $d$ for which $G$ has an affine representation is the *affine dimension* of $G$. If each set is a subspace of $k^d$, and two vertices are adjacent if and only if their corresponding subspaces have a nontrivial intersection, then $G$ has a *projective representation* in $k^d$ and the smallest such $d$ is the *projective dimension* of $G$ [6,7]. Pudlák and Rödl investigated the affine dimension and projective dimension of bipartite graphs arising from Boolean functions, and gave asymptotic bounds on these dimensions. They left the explicit construction of a graph with large affine or projective dimension as an open problem [6].

In this paper we consider graphs representable by $e$-dimensional affine subspaces of $\mathbb{R}^d$, where $e < d$. Though similar to both segment intersection graphs and graphs with an affine representation, these graphs have not been previously studied.

Formally, we say that a graph $G$ is a $(d, e)$-*subspace intersection graph* or $(d, e)$-*SI graph* if there exists a set of $e$-dimensional affine subspaces $R$ in $\mathbb{R}^d$ and a one-to-one correspondence between vertices in $G$ and subspaces in $R$, such that two vertices $v$ and $w$ in $G$ are adjacent if and only if their corresponding subspaces intersect. Note that since $R$ is a set, the subspaces are required to be distinct. For a given graph $G$, if such a set of subspaces $R$ exists, $R$ is called a $(d, e)$-*subspace intersection representation* or $(d, e)$-*SI representation* of $G$. For ease of reference, if $G$ is a $(d, e)$-SI graph with $(d, e)$-SI representation $R$, we denote the vertices of $G$ using lower-case letters, and the corresponding subspaces in $R$ using upper-case letters. For example, if $a$ and $b$ are vertices of $G$, then we denote their corresponding subspaces by $A$ and $B$.

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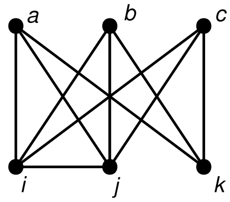


**Fig. 1.** The graph $G_2$.

In this paper, we seek to order the classes of $(d, e)$-SI graphs by set containment. Fig. 2 shows a partial order which summarizes our results. In this figure, edges represent set containment, and the graphs labeling the edges are separating examples.

**Lemma 1.** *Every $(d, e)$-SI graph is a $(d + k, e + j)$-SI graph, for all $k \geq j \geq 0$.*

**Proof.** Given a $(d, e)$-SI representation of a graph $G$, we can consider it in $(d + k)$-space, and use $j$ of these new dimensions to increase the dimension of each of the affine subspaces to $(e + j)$. □

Recall that a *complete multipartite graph* is a graph whose vertices can be partitioned into sets so that $u \sim v$ if and only if $u$ and $v$ belong to different sets of the partition.

**Proposition 2.** *A graph is a $(d, d - 1)$-SI graph if and only if it is a complete multipartite graph, for all $d \geq 2$.*

**Proof.** If $G$ is a $(d, d - 1)$-SI graph, then $G$ has a representation with $(d - 1)$-dimensional hyperplanes in $d$-space. Note that two hyperplanes are disjoint if and only if they are not parallel. Hence $G$ is a complete multipartite graph, with sets of vertices represented by parallel classes of hyperplanes forming the partite sets. Similarly, given a complete multipartite graph $G$, we can form a $(d, d - 1)$-SI representation of $G$ by taking parallel classes of hyperplanes in $d$-space. □

**Corollary 3.** *A graph is a $(d, d - 1)$-SI graph if and only if it is a $(2, 1)$-SI graph.* □

Let $G_1$ be the graph with vertices $a$, $b$, and $c$, and the single edge $\{a, b\}$. Note that $G_1$ cannot be represented as the intersection graph of lines in the plane, since line $C$ cannot be parallel with both line $A$ and line $B$. However, $G_1$ can be easily represented as the intersection graph of lines in 3-space. Hence we have the following proposition.

**Proposition 4.** *The graph $G_1$ is a $(3, 1)$-SI graph but not a $(2, 1)$-SI graph.*

## 2. Main results

Let $G_2$ be the graph formed by adding a single edge to the complete bipartite graph $K_{3,3}$. We label the vertices of $G_2$ as shown in Fig. 1.

**Theorem 5.** *The graph $G_2$ is a $(4, 2)$-SI graph but not a $(d, 1)$-SI graph for any $d \geq 2$.*

**Proof.** We present a $(4, 2)$-SI representation of $G_2$ by giving equations for the six planes in the representation. Using the convention that the four coordinates of $\mathbb{R}^4$ are $x$, $y$, $z$, and $w$, define these six planes to be $A : z = 1, w = 0$, $B : z = 2, w = 0$, $C : z = 3, w = 0$, $I : x = 1, y = z$, $J : x = 1, y = -z$, and $K : x = 0, w = 0$.

Now we prove that $G_2$ is not a $(d, 1)$-SI graph for any $d$. Suppose by way of contradiction that $R$ is a $(d, 1)$-SI representation of $G$ with lines $A$, $B$, $C$, $I$, $J$, and $K$. Since $I$ and $J$ intersect, they determine a plane $P$. Since $A$, $B$ and $C$ are mutually disjoint, at most one of them contains the intersection of $I$ and $J$. Hence two of them (without loss of generality $A$ and $B$) are both also contained in $P$. So $A$ and $B$ are parallel. Since $K$ intersects both $A$ and $B$, $K$ is also in $P$. Since the induced subgraph on the vertices $i$, $j$, and $k$ is isomorphic to $G_1$, this contradicts Proposition 4. Therefore $G_2$ is not a $(d, 1)$-SI graph for any $d \geq 2$. □

**Lemma 6.** *A graph $G$ is a $(d, e)$-SI graph if and only if it is a $(d - 1, e)$-SI graph for all $d > 2e + 1$.*

**Proof.** By Lemma 1, every $(d - 1, e)$-SI graph is also a $(d, e)$-SI graph. Conversely, suppose that $G$ is a $(d, e)$-SI graph with $n$ vertices, and $R$ is a $(d, e)$-SI representation of $G$ consisting of the set of affine subspaces $S_1, \ldots, S_n$ in $\mathbb{R}^d$. We construct a $(d - 1, e)$-SI representation $R'$ of $G$ by projecting $R$ onto a $(d - 1)$-dimensional subspace $V$ of $\mathbb{R}^d$. We prove that we may choose $V$ so that it has the following properties:

(1) $\dim(\mathrm{proj}_V(S_i)) = \dim(S_i)$ for all $1 \leq i \leq n$.
(2) For any two subspaces $S_i$ and $S_j$ in $R$, $\mathrm{proj}_V(S_i)$ and $\mathrm{proj}_V(S_j)$ intersect if and only if $S_i$ and $S_j$ intersect.

If we have found such a subspace $V$, then the new set of subspaces $\{\mathrm{proj}_V(S_i) \mid S_i \in R\}$ will be a $(d - 1, e)$-SI representation of $G$, and the proof will be complete.

Recall that, for a subspace $V$ in $\mathbb{R}^d$, the *orthogonal complement* $V^\perp$ is the subspace $\{\mathbf{v} \in \mathbb{R}^d \mid \mathbf{v} \perp V\}$. Every $(d - 1)$-dimensional subspace $V$ in $\mathbb{R}^d$ corresponds to a unique line through the origin $V^\perp$. We say that a projection onto $V$ is a projection *along* $V^\perp$. Also, given an affine subspace $S_i$ in $R$, let $S_i'$ be the unique subspace in $\mathbb{R}^d$ parallel with $S_i$ and passing through the origin. Note that $\dim(\mathrm{proj}_V(S_i)) = \dim(S_i)$ if and only if $V^\perp$ is not contained in $S_i'$.

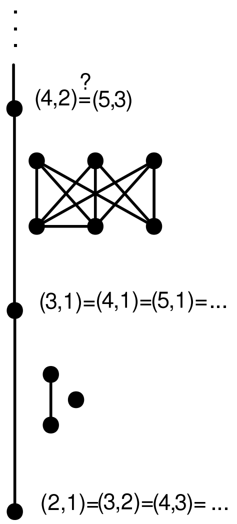


**Fig. 2.** The hierarchy of classes of $(d, e)$-SI graphs.

For two affine subspaces $S_i$ and $S_j$ in $R$, let $p_i$ and $p_j$ be points in $S_i$ and $S_j$, respectively. Given a $(d-1)$-dimensional subspace $V$ of $\mathbb{R}^d$, the projection onto $V$ maps $p_i$ and $p_j$ onto the same point in $V$ if and only if the projection is along the line spanned by the vector $p_i - p_j$. Given a pair of non-intersecting affine subspaces $S_i$ and $S_j$, consider the span of all such vectors, $A_{ij}$. Note that $A_{ij}$ is also the span of all vectors in $S_i'$ and $S_j'$ and a single vector of the form $p_i - p_j$. Since $\dim(S_i) = \dim(S_j) = e$, $\dim(A_{ij}) \leq 2e + 1$.

Since $R$ has a finite number of subspaces, and $d > 2e + 1$ by hypothesis, the union $\mathcal{S}$ of all the sets $A_{ij}$ and all the sets $S_i'$ cannot be all of $\mathbb{R}^d$. So there exists some other line $l$ through the origin, not contained in $\mathcal{S}$. Therefore the $(d-1)$-dimensional subspace $l^\perp$ is the required subspace. □

The following theorem follows directly from Lemma 6.

**Theorem 7.** *A graph $G$ is a $(d, e)$-SI graph if and only if it is a $(2e+1, e)$-SI graph for all $d \geq 2e+1$. In particular, $G$ is a $(d, 1)$-SI graph if and only if it is a $(3, 1)$-SI graph for all $d \geq 3$.* □

We conjecture that Theorem 7 can be strengthened as follows.

**Conjecture 1.** *A graph $G$ is an $(e+k, e)$-SI graph if and only if $G$ is an $(e+2, e)$-SI graph, for $k \geq 2$. In particular, we conjecture that if $G$ is a $(5, 2)$-SI graph then $G$ is a $(4, 2)$-SI graph.*

**Theorem 8.** *Given a finite graph $G$, there exist positive integers $d$ and $e$ such that $G$ is a $(d, e)$-SI graph.*

**Proof.** Suppose $G$ is a finite graph with vertices $v_1, \ldots, v_n$. Let $\mathcal{E} = \{e_1, \ldots, e_k\}$ be the set of all unordered pairs of vertices of $G$ (so $k = \binom{n}{2}$). Note that $E(G) \subseteq \mathcal{E}$.

We define the affine subspace $V_i \subseteq \mathbb{R}^k$ in the following way. Let $P_i$ be the set of positive integers $p$ such that $e_p$ contains $v_i$ and is not an edge of $G$. Then $V_i = \{\mathbf{x} \mid \mathbf{x}$ has an $i$ in coordinate $p$ for all $p \in P_i\}$. We modify the subspaces $V_i$ so that they all have the same dimension. So let $D$ be the largest dimension of any $V_i$, and $d$ be the smallest dimension of any $V_i$. For each $1 \leq i \leq n$ let $U_i$ be any affine subspace of $\mathbb{R}^{D-d}$ of dimension $D - \dim(V_i)$, and let $V_i' = V_i \times U_i$. We claim that the set $R = \{V_i'\}$ is a $(k + D - d, D)$-SI representation of $G$.

We must prove that two vertices $v_a$ and $v_b$ of $G$ are adjacent if and only if their corresponding subspaces $V_a'$ and $V_b'$ intersect. So first, suppose that $v_a \not\sim v_b$ in $G$, and $e_p = \{v_a, v_b\}$ in $\mathcal{E}$. Then the $p$th coordinate of every point in $V_a'$ is $a$ and the $p$th coordinate of every point in $V_b'$ is $b$. Thus $V_a' \cap V_b' = \varnothing$.

On the other hand, suppose that $v_a \sim v_b$ in $G$, and $p$ is a coordinate in which every point in $V_a'$ has the same value $x$. Note that $x$ must be either $a$ or 0 by the definition of $V_a'$. If $x = a$, then $e_p$ is a non-edge containing $v_a$. Since $v_a \sim v_b$, $e_p$ does not contain $v_b$, and so points in $V_b'$ can take on any value in their $p$th coordinate. If $x = 0$, then $p > k$, and so points in $V_b'$ can take on 0 as their $p$th coordinate. Hence we can find a point in both $V_a'$ and $V_b'$, so $V_a' \cap V_b' \neq \varnothing$. □

It is likely that the $(d, e)$-SI representation of $G$ constructed in the proof of Theorem 8 is not the smallest such representation with respect to either $d$ or $e$.

By Corollary 3, the classes of $(2, 1)$-SI graphs, $(3, 2)$-SI graphs, etc., are all the same class of graphs, and by Lemma 1, this class of graphs is contained in all other classes of $(d, e)$-SI graphs. Similarly by Theorem 7, the classes of $(3, 1)$-SI graphs, $(4, 1)$-SI graphs, etc., are all the same class of graphs, and by Lemma 1, this class of graphs is contained in all classes of

$(d, e)$-SI graphs other than the $(2, 1)$-SI graphs. Furthermore, if Conjecture 1 is true, then the class of $(2, 1)$-SI graphs and the classes of $(e+2, e)$-SI graphs are all of the distinct classes of $(d, e)$-SI graphs. Fig. 2 shows the first few classes of this hierarchy, which summarizes the theorems in this paper.

Recall that the affine dimension of $G$ is the smallest value of $d$ such that $G$ is representable as the intersection graph of affine subspaces of $d$-space. Since there is only one choice for $e$ if $d = 2$, the graphs with affine dimension 2 are precisely the complete multipartite graphs by Proposition 2. Pudlák and Rödl ask for explicit constructions of graphs with large affine dimension [6]. We have been unable to find a graph which is not a (4, 2)-SI graph, so we do not know a graph with affine dimension greater than 4.

**Open Question 1.** *Find a graph which is not a* $(4, 2)$*-SI graph. More generally, find a graph with large affine dimension.*

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