Yiannis Sakellaridis

NEW YORK NUMBER THEORY SEMINAR
Talk, 7 February 2008

1 Introduction

G: (connected) reductive algebraic group over $\begin{cases} k \text{ global} & \text{– automorphic reps : } \nu : \pi = \otimes'_v \pi_v \overset{\text{unitary}}{\hookrightarrow} L^2_{\text{disc}}(G(k)\backslash G(\mathbb{A})) \\ k \text{ local} & \text{– smooth, unitary etc. reps,} \end{cases}$
$H \subset G$ a "large" subgroup. We set $X = H\backslash G$.
Let us assume also that G is split and H is reductive.

This is the underlying set-up for a great deal of methods in the theory of automorphic forms, such as:

- Period integrals.
- (The relative trace formula – in the background of today's dicussion as a major source of examples but also as a basic method of proof.)
- ((The Rankin-Selberg method – will not discuss today, but can be put in the same context.))

Goal: to show that there is a general scheme underlying these methods, of which the study of representations of a group is just a special case (when we take $X = H, G = H \times H$).

Example 1.0.1. $G = \text{PGL}_2$, $H =$ split torus. $\nu : \pi \ni v \mapsto \phi \in L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A}))$.

$$\int_{[H]} \phi(h)dh \text{ “=” } L(\pi, \frac{1}{2}).$$

Philosophical reason: The period integral represents an element of $\text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) = \text{Hom}_{G(\mathbb{A})}(\pi, C^\infty(H(\mathbb{A})\backslash G(\mathbb{A}))$. We have $\text{Hom}_{G(\mathbb{A})}(\pi, C^\infty(H(\mathbb{A})\backslash G(\mathbb{A})) = \otimes'_v \text{Hom}_{G_v}(\pi_v, C^\infty(H_v\backslash G_v))$ with all the factors one-dimensional, therefore a global $H(\mathbb{A})$-invariant functional is a product of local functionals.

Example 1.0.2. $U =$ maximal unipotent subgroup. $\nu : \pi \to C^\infty(G(k)\backslash G(\mathbb{A}))$.

$$\int_{[H]}\Bigg|_{\nu(\pi)} \neq 0 \iff (\nu, \pi) \text{ is principal Eisenstein.}$$

Philosophical reason: Only representations with specific L-parameters can admit H-invariant functionals. In this case, only those with $\mathcal{L}_k \to A^* \subset \check{G}$. (This is not very precise globally – for instance, everywhere unramified cusp forms could admit $U(\mathbb{A})$-invariant functionals – yet it is true!)

Definition. We say that (ν, π) is *globally distinguished* if $\int_{[H]} \circ \nu \neq 0$.

We say that it is *locally distinguished* if $\text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$.

Example 1.0.3. $G = \text{Res}_{E/k} \text{GL}_n$, E/k: quadratic extension of number fields. $\nu : \pi \hookrightarrow L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A}))$.

Theorem ($\Rightarrow$ Oda, $\Leftarrow$ Jacquet (& Lapid)). *π is distinguished by* some *unitary subgroup H for E/k $\iff$ it is a base change from GL_n /k.*

Theorem (Lapid-Offen). *In the above setting, if π is a base change and H is the unitary group which distinguishes it then (under some additional assumptions) the period integral is related to some L-value.*

So, we are looking for a scheme of the following form:

$$\int_{[H]} \circ \nu = \begin{cases} 0, \text{ if the } L\text{-parameters of } (\nu, \pi) \text{ are not of a particular form (and } \pi \text{ is locally distinguished).} \\ \text{a specific } H\text{-invariant functional related to some } L\text{-value, otherwise.} \end{cases}$$

First we will examine the local question.

2 Locally distinguished representations.

For this section: k is local, non-archimedean, characteristic zero.

2.1 The dual group of a spherical variety.

$H \subset G$, $X = H\backslash G$. Spherical: The Borel $B \subset G$ has an open orbit $\mathring{X}$.

An X-*distinguished* representation π is $\nu : \pi \hookrightarrow C^\infty(X)$. (Difference between $X(k)$ and $H(k)\backslash G(k)$ will be mostly ignored for this talk.)

$\boxed{Q1 :}$ Describe distinguished representations.

$\boxed{Q2 :}$ Write a Plancherel formula: $L^2(X) = \int_{\hat{G}} \mathcal{H}_\pi d\mu(\pi)$.

(The two are closely related, will discuss.)

There will be a (partial) conjectural answer expressed in terms of a dual group $\check{G}_X \subset \check{G}$. The history of this:
- Knop: $X \leftrightarrow$ a root system with Weyl group W_X.
- Gaitsgory & Nadler: $X \leftrightarrow \check{G}_X \subset \check{G}$.

Remark. The maximal torus of $\check{G}_X$ has been identified, and in some cases its Weyl group (as W_X – conjecturally this is the answer). We have a conjectural description of $\check{G}_X$, and will henceforth assume it to hold.

Example 2.1.1. $X = H$, $G = H \times H$, $\check{G}_X = \check{H}^{\mathrm{diag}} \hookrightarrow \check{G}$.

Example 2.1.2. $X = U\backslash G$, $\check{G}_X = A^*$ (the maximal torus of $\check{G}$).

Example 2.1.3. $X = \mathrm{GL}_n \times \mathrm{GL}_n \backslash \mathrm{GL}_{2n}$, $\check{G}_X = \mathrm{Sp}_{2n}$.

2.2 The unramified spectrum.

Fix a "good" maximal compact subgroup K, define "unramified" with respect to K. Recall:

$$\begin{array}{ccc} \{\text{is. cl. of irreducible unramified representations of } G\} & \leftrightarrow & \{\text{ semisimple conjugacy classes in } \check{G}\} = A^*/W \\ \ni & & \ni \\ \pi & \leftrightarrow & \text{(unramified subquotient of) } I_B^G(\chi). \end{array}$$

Definition: $I_P^G(\sigma) = \mathrm{Ind}_P^G(\sigma\delta_P^{\frac{1}{2}})$.

Theorem 2.2.1 (S.). *There is an "almost bijection":*

$$\begin{array}{ccc} \{\textit{A basis for } \nu : \pi \textit{ (unramified irrep.)} \hookrightarrow C^\infty(X)\} & < -- > & \{\textit{ semisimple conjugacy classes in } \check{G}_X\} = A_X^*/W_X \\ \ni & & \ni \\ \textit{(explicit intertwining operators)} & & I_{P(X)}^G(\chi) \end{array}$$

Definition. $P(X) = \{g \in G | \mathring{X}g = \mathring{X}\}$.

Example 2.2.2. $P(X) = B$ in the cases $X = H, U\backslash G$, but in the case $\mathrm{Sp}_{2n} \backslash \mathrm{GL}_{2n}$ it is the $(2, 2, \dots, 2)$ parabolic.

Remark. The theorem accounts for some of the multiplicity, for instance in the $U\backslash G$ case there is W-fold multiplicity, which accounts for the isomorphisms between principal series.

However, it was stated imprecisely – one should in general introduce a multiplicity factor m = number of $B(k)$-orbits on $\mathring{X}(k)$. This suggests that $\check{G}_X$ is not the correct dual group, but a cover thereof: $\tilde{G}_X \twoheadrightarrow \check{G}_X$. In some cases, such a group exists; in others (e.g. $O_2\backslash \mathrm{GL}_2$) it doesn't, and the consensus in geometric Langlands seems to be that there should be something like a quantum group instead. In any case, $m = 1$ in the multiplicity-free case.

Theorem 2.2.3 (S.).[1] *Plancherel formula for* $L^2(X)^K$*:*

$$\|\Phi\|^2 = \int_{A_X^*/W_X} |\langle \Phi, \Omega_\chi \rangle|^2 d\chi$$

---

[1] Underlined theorems mean that the proof is complete up to establishing certain combinatorial features of the spherical variety. (In each separate case, this can be done easily.)

where $\Omega_\chi \in \nu(I^G_{P(X)}(\chi))^K$.

Remarks.
1. More information: Explicit (Casselman-Shalika) formula for the Hecke eigenfunctions Ω_χ.
2. Other results: More precise version of theorem 2.2.1, theorem on "good test vectors".
3. Uses a criterion of Bernstein which we will discuss later.

2.3 Arthur parameters.

First, recall Langlands parameters:

$$\phi : WD_k \to \check{G}.$$

Would like to say that distinguished representations have Langlands parameters with image in $\check{G}_X$, except that this is already false for unramified representations.

Let us turn our attention to unitary representations. If ϕ is a Langlands parameter with bounded image, then all representations in Π_ϕ (the L-packet) should be tempered, i.e. in $L^2(G)$. Which others are unitary? Arthur: "Not all unitary representations matter for arithmetic, but only some of them" such as the trivial repr. To those, he associated (conjectural) A-parameters:

$$\psi : WD_k \times \mathrm{SL}_2(\mathbb{C}) \to \check{G}$$

always assumed with bounded image.

For example: If χ is a unitary unramified character of P, then:

$$\text{(the unramified quotient of) } I^G_P(\chi) \quad \text{is Arthur with parameter} \quad \psi : \mathrm{Frob} \times \mathrm{SL}_2 \mapsto \chi \times \text{ principal } \mathrm{SL}_2 \text{ of } \check{P} \subset \check{G}.$$

Remark. In general, A-packets contain multiple L-packets and are not mutually disjoint. However, for unramified representations there are no such issues, and the L-parameter can easily be read off from the A-parameter.

2.4 The local conjecture.

In parts of the results stated below, we need to assume that X is *wavefront*. This includes symmetric varieties but not, for instance, the variety $U\backslash G$ (although the conjectural picture should hold for all).

Theorem 2.4.1 (S.- Venkatesh). *Assuming that $\check{G}_X$ (the Gaitsgory-Nadler group) is what we expect it to be, it commutes with the image of a principal SL_2 of $\check{L}(X)$.*

The theorem uses the fact that the Gaitsgory-Nadler group commutes with $\check{\rho}_{\check{L}(X)}$ (the $\mathbb{G}_m$-part of the principal SL_2), which is implicit in their paper. Hence we have $\check{G}_X \times \mathrm{pr}_{\check{L}(X)}(\mathrm{SL}_2) \subset \check{G}$.

Conjecture 2.4.2 (S.- Venkatesh). *The support of $L^2(X)$ is contained in the set of representations with Arthur parameter ψ such that $\psi(WD_k)$ is contained in $\check{G}_X$ and $\psi|_{\mathrm{SL}_2}$ is equal to $\mathrm{pr}_{\check{L}(X)}$.*

There is global-to-local motivation for this conjecture (partly an argument of Clozel).

Remark. A Plancherel formula contains information about "which representations (generically) are distinguished". For instance:

Proposition 2.4.3. *If π is a discrete series on G and it is X-distinguished, then it appears with positive measure in the Plancherel formula.*

There is a more precise way to formulate the conjecture, for instance Arthur parameters come with natural measures on them, and these should correspond to Plancherel measures. Using a criterion of Bernstein we prove:

Theorem 2.4.4 (S.- Venkatesh; based on an argument of Bernstein). *The space $L^2(X)$ admits a canonical direct sum decomposition $L^2(X) = \oplus_{\Theta/\sim} L^2(X)_\Theta$ where the $L^2(X)_\Theta$ are explictly described in terms of the discrete spectrum of certain spherical varieties X_Θ for certain "relevant" Levi subgroups L_Θ.*

For example, in the group case ($X = H, G = H \times H$) one recovers Harish-Chandra's Plancherel formula, where the summands are in terms of parabolically induced discrete series.

In the course of our proof we also prove:

Theorem 2.4.5 (S.- Venkatesh). *For every irreducible representation π, $\dim \mathrm{Hom}(\pi, C^\infty(X)) < \infty$.*

3 Global period integrals and the Ichino-Ikeda conjectures.

3.1 The Gross-Prasad conjecture after Ichino-Ikeda-Waldspurger.

$G = \mathrm{SO}_{n+1} \times \mathrm{SO}_n$, $H = \mathrm{SO}_n^{\mathrm{diag}}$. Endow G and H with Tamagawa measure. Let $\nu : \pi \hookrightarrow L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ denote a tempered cuspidal representation.

Conjecture 3.1.1 (Ichino-Ikeda(-Waldspurger)). *For $u = \otimes u_v \in \pi$ with $\phi = \nu(u)$ we have:*

$$\left| \int_{[H]} \phi(h) dh \right|^2 =? \cdot \text{“} \prod_v \int_{H_v} \langle h \cdot u_v, u_v \rangle \, dh \text{”}. \tag{1}$$

Remarks. 1. The factor $?$ has to do with sizes of A-packets, I don't understand it and I will ignore it in the discussion that follows.

2. While I-I show that the integral for each place v converges absolutely, the product does not, and one has to understand it in the sense of analytic continuation. More precisely, they show that if u_v is the “standard” unramified vector of π_v, then $\int_{H_v} \langle h \cdot u_v, u_v \rangle \, dh = L(\pi_v)$, where $L(\pi_v)$ is a certain quotient of L-values for π. Therefore, the rigorous expression is:

$$\left| \int_{[H]} \phi(h) dh \right|^2 =? \cdot L^S(\pi) \prod_{v \in S} \int_{H_v} \langle h \cdot u_v, u_v \rangle \, dh \tag{2}$$

where S is a sufficiently large finite set of places and L^S is the corresponding partial L-value.

3.2 The local I-I conjecture.

(Back to local field notation.) The local conjecture of I-I says that if π is tempered and H-distinguished then the skew-symmetric form $\Lambda_\pi : (v, u) \mapsto \int_H \langle h \cdot v, u \rangle$ is positive definite. (Non-negativity was taken care of in the final version of their paper, so what remains to be shown is that it is non-zero.)

We make the following observation: Let $f_i \in C_c^\infty(G)$ and $\Phi_i(x) = \int_H f_i(hx) dh \in C_c^\infty(X)$, $i = 1, 2$. From the “standard” Plancherel formula for G:

$$\langle f_1, f_2 \rangle_{L^2(G)} = \int_{\hat{G}} C M_\pi^*(f_1, f_2) d\mu_{\mathrm{Planch}}(\pi)$$

expressed in terms of matrix coefficients M_π, we deduce the following Plancherel formula for X:

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\hat{G}} C \Lambda_\pi^*(\Phi_1, \Phi_2) d\mu_{\mathrm{Planch}}(\pi)$$

where Λ_π is the skew-symmetric pairing of above!

Remark. A Plancherel formula consists of a measure on $\hat{G}$ (the unitary dual) together with a family of morphisms: $\Lambda_\pi : \pi \otimes \bar{\pi} \to C^\infty(X) \otimes C^\infty(X)$ such that the adjoint:

$$C_c^\infty(X) \otimes C_c^\infty(X) \xrightarrow{\Lambda_\pi^*} \pi \otimes \bar{\pi} \xrightarrow{C} \mathbb{C}$$

is positive definite. In the above, M_π denotes the morphism corresponding to matrix coefficients.

This immediately implies that the form Λ_π is non-negative for almost every π and, by continuity, for every π. By analyzing more carefully abstract properties that Λ_π should have we prove:

Theorem 3.2.1 (S.- Venkatesh). *For $d\mu_{\mathrm{Planch}}(\pi)$-almost every π, if π is H-distinguished then Λ_π is non-zero. If G is split, the same hold for every π.*

Remarks. 1. The proof for “every” is based on Bernstein's criterion and should work in general.

2. For proper subrepresentations of $I_P^G(\sigma)$, where σ is a discrete series, our theorem is conditional on “multiplicity one for $I_P^G(\sigma)$”, a special case of the Gross-Prasad “multiplicity one in L-packets” conjecture.

3.3 Global period integrals.

One can attempt to generalize the global I-I-W conjecture to every spherical Gelfand pair $G \supset H$ (H reductive). More precisely, if our local conjecture for the support of Plancherel measure is correct, then one can write a Plancherel formula for X using the "standard Plancherel measure" for G_X (a group whose dual is $\check{G}_X$); let $\Lambda_\pi : \pi \otimes \bar{\pi} \to C^\infty(X \times X)$ be the forms (possibly zero or infinite, but conjecturally not – on the distinguished spectrum!) corresponding to such a choice of measure

One could then conjecture, up to factors ? which we don't understand, that:

$$\left| \int_{[H]} \phi \right|^2 = ? \cdot \text{“} \prod_v \Lambda_{\pi_v}(u_v \otimes \bar{u}_v) \text{”}. \tag{3}$$

(where $\phi = \nu(u)$) at least for $\nu : \pi \hookrightarrow L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A}))$ with Arthur parameter $\psi|_{\mathrm{SL}_2} = \mathrm{pr}_{\check{L}(X)}$.

Remark. Again, the infinite product would only make sense as an L-value. Indeed:

Theorem 3.3.1 (S.). *If $G \supset H$ is a spherical Gelfand pair with G split and H reductive (over a p-adic field), and if $\pi \in L^2(X)$ is unramified, Λ_π as above and $u \in \pi^K$, then:*

$$\frac{\Lambda_\pi(u \otimes \bar{u})}{\|u\|^2} = \textit{some explicit } L\textit{-value.}$$

It is interesting to remark that the conjecture (3) is non-trivial even in the group case: $X = H$, $G = H \times H$. In that case the left side is equal to 1 and the right side is a product of "formal dimensions" (with respect to "Haar" Plancherel measure), most of which are known to be equal to certain adjoint γ-factors ($\gamma(0, \pi, \mathrm{Ad})$). Thus by the fact that the global gamma factor is one, and up to the factors ?, we retrieve a conjecture of Hiraga, Ichino and Ikeda, associating formal dimensions to γ-factors.

4 The wavefront lemma, the Bernstein criterion and some local proofs.

4.1 Cartan decomposition, wavefront lemma and finiteness.

$G \supset B \supset A$ split, $K = G(\mathfrak{o})$ hyperspecial, $x_0 \in X(\mathfrak{o})$, $A_X := x_0 \cdot A$.

Theorem 4.1.1 (Generalized Cartan decomposition). *If G, X are defined globally (G split) then at almost every place $X = A_X^+ K$. (In general, a finite number of translates of $A_X^+ K$.)*

Remarks.

1. Proven by Luna-Vust for $k = \mathbb{C}((t))$, reproven by Gaitsgory and Nadler. Proof of Gaitsgory and Nadler essentially works in p-adic setting.
2. For symmetric varieties, a version of this was proven in more generality by Delorme-Sescherre and Benoist-Oh.

Definition: X is "wavefront" if $A^+ \twoheadrightarrow A_X^+$. From now on, assume for simplicity that $G(k) \twoheadrightarrow X(k)$.

Lemma 4.1.2 (The wavefront lemma). *Let π be a representation of G, $L : \pi \xrightarrow{H} \mathbb{C}$. For every open compact subgroup J_1 there exists an open compact subgroup J_2 such that for every $u \in \pi^{J_1}$ and every $g \in A^+K$ we have:*

$$\langle L, \pi(g)u \rangle = \langle J_2 * L, \pi(g)u \rangle .$$

Hence if the variety is wavefront, an H-invariant functional can be analyzed with the help of smooth functionals $J_2 * L$. For instance, if $L \leftrightarrow \nu : \pi \to C^\infty(X)$ and a non-zero $u \in \pi^{J_1}$ with $\phi := \nu(u)$ we have $\phi(x) = \langle J_2 * L, \pi(g)u \rangle$ where $g \in A^+K, x_0 g = x$, and since $J_2 * L \in \tilde{\pi}^{J_2}$, which is finite dimensional, this implies our theorem that $\dim \mathrm{Hom}(\pi, C^\infty(X)) < \infty$.

4.2 Bernstein's criterion.

We turn to the Plancherel formula. Let us describe Bernstein's criterion for the case $X = H = \mathrm{PGL}_2$, $G = H \times H$. Here we have the Cartan decomposition: $G = KA^+K = K\mathbb{Z}_+K$. Suppose given a Plancherel formula:

$$\|f\|^2 = \int_{\hat{G}} \cdots d\mu(\pi)$$

where $\cdots$ is given in terms of inner products of f with matrix coefficients of π. We take f supported in a small neighborhood of A^+ and far away from the origin. From the asymptotics of matrix coefficients in terms of Jacquet modules we know:

> All matrix coefficients of π eventually are linear combinations of terms of the form $\chi(a)$, where χ is a character of A.

Here we distinguish two cases for π: Either all characters χ appearing are "quickly decreasing" in the $\mathbb{Z}_+$ direction, in which case π is a discrete series; or they are not, in which case they will have a significant contribution to the L^2 norm of f no matter how far we take f from the origin.

The point now is that taking f far from the origin, the Plancherel formula for f starts looking like Plancherel formula for A. Therefore, apart from discrete series (this method gives no information about formal degrees of those), the contribution of π's with non-decreasing asymptotics should be weighed according to the (known) Plancherel formula for A.

In the general case, there is an inductive process involving a set of "relevant parabolics" P_Θ and Jacquet modules, and we build up $L^2(X)$ as *discrete part + pieces understood via the discrete parts of varieties* X_Θ *(spherical for* L_Θ*)*.

The non-zero conjecture of Ichino and Ikeda follows from the fact that the morphisms Λ_π should, by Bernstein's criterion, have non-zero asymptotics (for a.e. π and, by continuity, every π).

Remark. Bernstein's criterion is much more general – e.g. can be used in the automorphic case – and can be used to obtain very non-trivial results such as the Maaß-Selberg relations.

5 Epilogue.

H-period integrals and distinguished representations have attracted the attention of a great part of the "automorphic forms" community, but it is probably fair to say that they have also been ignored by another great part. The reason is probably that there are a lot of interesting phenomena but they appear disparate and special, with no general picture. On the contrary, though, it seems to me that a general picture exists, and hopefully this talk helped convince you about this. It seems that one should be able to even understand "strange" phenomena such as the factorization of unitary period integrals, where there is no multiplicity one, in terms of the "dual data" of the spherical variety. Finally, I am developing a similar picture for subgroups H that are not reductive, together with an understanding of the Rankin-Selberg method, which will probably also help understand certain examples of the relative trace formula. All this suggests a fascinating general picture, of which the "group case" is only a special case!