Lecture 1[1]

1.1.1 References

- A. Borel, *Linear Algebraic Groups.* (2nd ed., Springer).
- T. A. Springer, *Linear Algebraic Groups.* (2nd ed., Birkhäuser).
- T. A. Springer, *Reductive Groups.* Automorphic forms, representations and L-functions, PSPUM 33.1, AMS (the "Corvallis proceedings").

1.1.2 Basic notions

Let k be a field.

A *group scheme* is a group in the category of k-schemes. Facts: If k is of characteristic zero, then any group scheme is *smooth* over k. In characteristic $p > 0$, though, a group scheme could be non-reduced, for instance the *Frobenius* group scheme, i.e. the kernel of $\mathbb{G}_m \to \mathbb{G}_m : g \mapsto g^p$. An *algebraic group* is a smooth group scheme over a field k. FROM NOW ON, ONLY CHARACTERISTIC ZERO.

The geometrically connected component G^0 of G is always defined over k and normal. (The group of components G/G^0 is also an algebraic group over k, cf. "homogeneous spaces".)

The *multiplicative group* $\mathbb{G}_m = \operatorname{spec} k[T, T^{-1}], T \mapsto T_1 \cdot T_2$. The *additive group* $\mathbb{G}_a = \operatorname{spec} k[T], T \mapsto T_1 + T_2$.

A *linear algebraic group* is one which is affine. Equivalently, it admits a closed embedding into GL_n for some n.

The *Jordan decomposition*: Every element $g \in \mathrm{GL}_n$ can be written uniquely as $g_s g_u$, with g_s *semisimple* and g_u *unipotent*, so that g_s and g_u commute. If $g \in \mathrm{GL}_n(k)$, then so are g_s and g_u. Let G be a linear algebraic group and $i : G \hookrightarrow \mathrm{GL}_n$ an embedding. Let $g \in G$. Facts: whether $i(g)$ is semisimple or unipotent does not depend on the embedding! Therefore g will be called semisimple or unipotent accordingly. Moreover, $g = g_s g_u$ uniquely, with $g_s \in G$ semisimple and $g_u \in G$ unipotent.

From now on "group" will mean "linear algebraic group" (over k).

1.1.3 Homogeneous spaces

If G is a group and H a closed subgroup, then $H\backslash G$, the *geometric quotient* of G by H, exists in the category of k-schemes, and is also smooth over k. (The existence statement is valid in positive characteristic as well, but of course the quotient may not be reduced if G is not.) However, *it quasi-projective but not necessarily affine.*

(Definition of geometric quotient $\pi : X \to Y$ of a scheme X by the action of G: π is open and surjective, $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$ and the geometric fibers are

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precisely the geometric orbits of G. For more details see, for instance, Mumford, *Geometric Invariant Theory* or the fourth chapter in the book of van der Geer and Moonen on abelian varieties, appearing in preliminary form at: `http://staff.science.uva.nl/~bmoonen/boek/BookAV.html`.)

The idea of the proof: Show that there exists an algebraic representation V of G such that H is the stabilizer of a line in V. One finds such a representation by playing with the ring of regular functions on G and the ideal defining H. Then $H\backslash G$ can naturally be identified with the orbit of that line in $\mathbb{P}(V)$. (General facts about algebraic representations and actions of G on varieties: By definition, their matrix coefficients are regular functions on G. Every action of G on an affine variety X is locally finite, i.e. every $f \in k[X]$ generates a finite-dimensional subspace. Every orbit of G on X is locally closed.)

Example : $P\backslash G$, where P is a parabolic subgroup, is projective. (This is actually the definition of *parabolic subgroup*, so the statement is void unless you know some examples of parabolic subgroups!) In fact, for reductive G the quotient $H\backslash G$ is affine if and only if H is also reductive. (Borel & Harish-Chandra.) We will explain these notions later.

If H is normal, then $H\backslash G$ has a natural group structure. Beware: the points of $(H\backslash G)(k)$ are not the same as $H(k)\backslash G(k)$, in general. Example: the space $O_n\backslash \mathrm{GL}_n$ of non-degenerate quadratic forms in n-variables.

1.1.4 Diagonalizable groups

The character group $\mathcal{X}(G)$ of G is the group of morphisms: $G \to \mathbb{G}_m$. If F is a superfield of the field of definition, we will write: $\mathcal{X}_F(G)$ for $\mathcal{X}(G_F)$ (where G_F denotes the base change to F), i.e. those characters defined over F.

Exercise. The only character of $\mathbb{G}_a$ is the trivial one.

(This has the following generalization: The only irreducible algebraic representation of a unipotent group is the trivial one.)

On the other hand, the characters of $\mathbb{G}_m$ are precisely those of the form $T \mapsto T^r$ for some $r \in \mathbb{Z}$. We notice that if we consider them as functions $\mathbb{G}_m \mapsto \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ then *every element of* $k[\mathbb{G}_m]$ *can be written as a linear combination of characters.* (In other words, $k[\mathbb{G}_m]$ is the Grothendieck ring over k of the group $\mathcal{X}_{\bar{k}}(\mathbb{G}_m)$.)

A *diagonalizable group* is one for which its $\bar{k}$-character group $\mathcal{X}_{\bar{k}}(G)$ spans $\bar{k}[G]$ over $\bar{k}$. Equivalently, it is commutative and semisimple. It is called *split* according as $\mathcal{X}_k(G) = \mathcal{X}_{\bar{k}}(G)$ or not. A *torus* is a group which is isomorphic, over $\bar{k}$, to $\mathbb{G}_m^r$ for some r. The connected component of a smooth diagonalizable group is a torus.

Theorem 1.1.1. *The contravariant functor $G \mapsto \mathcal{X}_{\bar{k}}(G)$ is an equivalence between the categories of diagonalizable k-groups and of finitely generated $\mathbb{Z}$-modules with a (continuous)* $\mathrm{Gal}(\bar{k}/k)$*-action.*

This is a relief: Everything that we would like to know about these groups can be reduced to a relatively simple combinatorial picture. Let us see an example:

Example 1.1.2. The character group of $\mathbb{G}_m^2$ is isomorphic to $\mathbb{Z}^2$; fix such an isomorphism. Let E be a quadratic extension of k. There is a diagonalizable group R which is isomorphic to $\mathbb{G}_m^2$ over the algebraic closure, but such that

$R(k) = E^\times$.[2] Let σ be the non-trivial element of $\mathrm{Gal}(E/k)$. Then we define the action of $\mathrm{Gal}(\bar{k}/k)$ on $\mathcal{X}(\mathbb{G}_m^2)$ as follows: It will factor through $\mathrm{Gal}(E/k)$, and ${}^\sigma(a,b) = (b,a) \in \mathbb{Z}^2$. The short exact sequence:

$$0 \to \{(a,-a)|a \in \mathbb{Z}\} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

(where $\mathrm{Gal}(\bar{k}/k)$ acts trivially on the last group) corresponds, dually, to a sequence of the groups:

$$1 \leftarrow R/\mathbb{G}_m \leftarrow R \leftarrow \mathbb{G}_m \leftarrow 1$$

which at the level of points is the embedding $k^\times \hookrightarrow E^\times$.

On the other hand there is another short exact sequence:

$$0 \to \{(a,a)|a \in \mathbb{Z}\} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

where now the Galois group acts trivially on the first term, but not on the last. This corresponds to the norm map:

$$E^\times \to k^\times.$$

Notice that the kernel of the norm map is an 1-dimensional torus. Moreover, notice that here we have a surjection of algebraic groups which is not a surjection at the level of k-points.

Remark. The group constructed above is called the *restriction of scalars* of $\mathbb{G}_m$ from E to k; namely, we start with the group $\mathbb{G}_m$ over E and construct an algebraic group R over k such that for every extension F of k we have $R(F) = \mathbb{G}_m(F \otimes E)$. We will return to this concept in a more general setting.

Exercise. Verify the above-mentioned property of restriction of scalars for the given example.

Exercise. Prove the following: If k is algebraically closed and D is a diagonalizable group, then there exists a finite subgroup F of D such that $D = D^0 \times F$. Does that hold for arbitrary k?

1.1.5 Reductive groups

A group is *solvable* if it admits a normal series whose successive quotients are abelian. Notice that this notion is k-independent, since we can take the normal series where G_i is followed by $[G_i, G_i]$, which is defined over k. A *Borel subgroup* of a linear group G is one which is maximal (over the algebraic closure) among the connected solvable subgroups. It is not true in general that there is a Borel subgroup over k; if there is, the group G will be called *quasi-split.* We will see later that every reductive group is an *inner form* of precisely one (isomorphism class of) quasi-split group.

Facts: All Borel subgroups are $G(k)$-conjugate. A subgroup is parabolic if and only if it contains a Borel. (The proof of these facts relies on (a generalization of) the Lie-Kolchin Theorem, which states that a connected solvable group acting on a proper variety has a fixed point.)

[2] We recall here the notion of *points* of a scheme: Given schemes X, Y over a basis S, we define $X(Y) := \mathrm{Mor}(Y, X)$ in the category of S-schemes (the "Y-points of X"). When $Y = \operatorname{spec} A$, for a ring A, we also say "A-points". If $S = \operatorname{spec} k$, X is affine and A is any k-algebra then $X(A) = \mathrm{Hom}(k[X], A)$. Of course, for our discussion, $G(A)$ is the same as $i(G) \cap \mathrm{GL}_n(A)$ for any closed embedding $i : G \to \mathrm{GL}_n$.

Every group G has a maximal closed, connected, normal, solvable subgroup. This is the *radical* $\mathcal{R}(G)$. The unipotent elements of $\mathcal{R}(G)$ form a maximal closed, connected, unipotent subgroup of G, the *unipotent radical* $\mathcal{R}_u(G)$. We say that G is *reductive* if $\mathcal{R}_u(G) = 1$ and *semisimple* if $\mathcal{R}(G) = 1$. The *rank* of G is the dimension of a maximal torus and its *semisimple rank* is the rank of its derived group; equivalently, the dimension of its root system (see below).

(Discussion of examples in class.)

Reductivity theorem: If G is reductive then every algebraic representation of G is completely reducible. This theorem does not hold in positive characteristic, where it is replaced by *geometric reductivity* (Haboush). (The latter states that for every G-invariant vector there is a G-invariant function, homogeneous of degree a power of the characteristic exponent of the field, which is non-zero on the given vector. If the characteristic exponent is 1, this leads to complete reducibility.)

The notion of semisimplicity is better understood at the level of Lie algebras. *Lie algebra* $\mathfrak{g}$ of G = tangent space at the identity = derivations of $k[G]$ in $k[G]$ invariant by left translation = differential operators on G homogeneous of degree 1 invariant by left translation. All the definitions given above (solvable, reductive, semisimple etc.) have obvious analogs for Lie algebras and a (connected) group G is ... if its Lie algebra is ... .

A Lie algebra $\mathfrak{g}$ is called *simple* if it is not abelian and its only ideals are 0 and $\mathfrak{g}$. Then: $\mathfrak{g}$ is semisimple if and only if it is isomorphic to a product of simple algebras. This is equivalent to There is also another criterion for when a Lie algebra is semisimple, sometimes taken as the definition: $\mathfrak{g}$ is semisimple iff the invariant bilinear form: $(X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ (the *Killing form*) is non-degenerate. A very nice reference for semisimple Lie algebras is the book of Serre: *Complex Semisimple Lie Algebras*.

1.1.6 Root systems and root data

For this section we assume k to be algebraically closed. In the next lecture we will discuss rationality issues.

We saw a nice combinatorial description for diagonalizable groups, we would have liked to have the same for more general classes of groups. This is not possible in the sense of getting an equivalence of categories[3], but at least we can fully describe the isomorphism classes of groups this way.

More precisely, the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ of a reductive group gives rise to its root datum: This is a combinatorial description of the structure of G.

Let G be reductive, and let A be a maximal torus in G. We denote by small gothic letters the corresponding Lie algebras. The group A is equal to its centralizer; the centralizer of a maximal torus in a (not necessarily reductive) algebraic group is called a *Cartan subgroup*. All Cartan subgroups are conjugate. Under the adjoint representation, $\mathfrak{g}$ decomposes into eigenspaces for A:

$$\mathfrak{g} = \mathfrak{a} \oplus \sum_{\alpha \in \Phi} \mathfrak{u}_\alpha.$$

[3] There is a good reason for it: To get an equivalence of categories one must consider the category of all G-representations, cf. Tannaka-Krein duality. For diagonalizable groups this category is described easily in terms of combinatorial data, this is no longer the case for other groups.

Facts: $\mathfrak{a}$ is precisely equal to the zero eigenspace (i.e. $\mathfrak{a}$ is its own commutator). The non-zero eigencharacters of A are called *roots* (their set denoted by Φ), and their eigenspaces $\mathfrak{u}_\alpha$ are all one-dimensional.

Recall that we study diagonalizable groups via their character lattices. The character lattice $\mathcal{X}(A)$ is free, and Φ is a finite subset of it. The *Weyl group* $W := \mathcal{N}(A)/A$ acts on $\mathcal{X}(A)$ and the set of roots. It is a finite group.

Let us first consider the case of SL_2: Its Lie algebra is generated by three elements $h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $e = \begin{pmatrix} & 1 \\ & \end{pmatrix}$ and $f = \begin{pmatrix} & \\ 1 & \end{pmatrix}$ which satisfy the commutation relations $[h,e] = 2e, [h,f] = -2f$ and $[e,f] = h$. Three non-zero elements (h,e,f) in a Lie algebra which satisfy these commutation relations are called an *$\mathfrak{sl}_2$-triple*. Here we have $\mathcal{X}(A) \simeq \mathbb{Z}$, with roots $\pm\alpha = \pm 2$ under such an isomorphism and $W = \mathbb{Z}/2$ which sends α to $-\alpha$.

Similarly, for every reductive group if $\alpha \in \Phi$ then $-\alpha \in \Phi$ and there are $h \in \mathfrak{a}$, $e \in \mathfrak{u}_\alpha$ and $f \in \mathfrak{u}_{-\alpha}$ such that (h,e,f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. In fact, h is unique, e can be chosen to be an arbitrary non-zero element and f is unique once e is chosen. The element h lives in $\mathcal{X}(A)^* := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}(A), \mathbb{Z}) = \mathrm{Hom}(\mathbb{G}_m, A)$, and it is called the *co-root* $\check{\alpha}$ associated to α. Hence, to the group G one can also associate the set $\check{\Phi}$ of co-roots, a finite subset of $\mathcal{X}(A)^*$.

1.1.6.1 Root systems

Both the sets of roots and of co-roots, together with the action of the Weyl group, each satisfy the axioms for a combinatorial object called *the root system*. Strictly speaking, a root system is formed by Φ and its linear span V in $\mathcal{X}(A) \otimes \mathbb{R}$ (and similarly for the co-roots). These axioms are:

1. Φ is a finite subset spanning V and not containing 0.
2. For every $\alpha \in \Phi$ there is a symmetry w_α (that is, a linear automorphism of V whose fixed point set is a hyperplane) such that $w_\alpha(\Phi) = \Phi$ and $w_\alpha(\alpha) = -\alpha$.
3. For each $\alpha, \beta \in \Phi$, $w_\alpha(\beta) - \beta$ is an integer multiple of α.

Moreover, the ("absolute") root system arising from a reductive group is *reduced*, or *crystallographic*: If $\alpha, c\alpha \in \Phi$ then $c = \pm 1$.

Now the reader should consult a reference on root systems (for instance, Serre's book) to learn more about them, their classification, etc. In short: Since W is a finite group, it is possible to find an inner product on V such that the w_α's are reflections. This inner product is unique up to multiple on every irreducible component (a root system is irreducible if it is not the sum of two sub-root systems). With respect to this inner product, for every $\alpha, \beta \in \Phi$ we have $w_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$. The angle between α and β, for α and β simple roots (for a definition of simple roots, see the next lecture), encodes the way in which w_α and w_β commute (more precisely, W is the group with generators w_α, with α ranging over simple roots, and relations $w_\alpha^2 = 1$ and $(w_\alpha w_\beta)^{m(\alpha,\beta)} = 1$ where $m(\alpha,\beta) = 2, 3, 4$ or 6 according as the angle between α and β is $\pi/2, 2\pi/3, 3\pi/4$, or $5\pi/6$). It also encodes relations for a set of generators of the Lie algebra $\mathfrak{g}$. The simple roots are the vertices of the *Dynkin diagram*, with an edge between two roots iff they are not orthogonal – the edge being simple, double or triple according as the angle between them is $2\pi/3, 3\pi/4$, or $5\pi/6$, and an arrow from the longest to the shortest, if they don't have the same length. The reduced, irreducible root systems have been classified: there are four infinite families A_n, B_n, C_n, D_n and five *exceptional root systems* G_2, F_4, E_6, E_7, E_8.

(Examples in class.)

Remark. The (hidden) condition that the Weyl group be finite imposes strong combinatorial restrictions that lead to only this finite number of families. There is a more general world, set up in a way that does not include the condition of finiteness, that of *Kac-Moody algebras.*

1.1.6.2 Root data

Notice that the root system only determines the adjoint group of G, for instance the groups GL_n, SL_n, PGL_n all have the same root system. We must find a way to encode information about the center of the group.

The data $(X := \mathcal{X}(A), \Phi, \check{X} := \mathcal{X}(A)^*, \check{\Phi})$ satisfy the axioms for a *root datum.* These consist of:

1. Two lattices (free $\mathbb{Z}$-modules of finite type) which are each other's dual.

2. Finite subsets $\Phi \subset X, \check{\Phi} \subset \check{X}$ with a bijection $\Phi \to \check{\Phi} : \alpha \mapsto \check{\alpha}$ such that $\langle \alpha, \check{\alpha} \rangle = 2$ (to understand this condition, recall the case of $\mathfrak{sl}_2$).

3. The endomorphisms of $X, \check{X}$ defined by $w_\alpha(x) := x - \langle x, \check{\alpha} \rangle \alpha$, $w_{\check{\alpha}}(\check{x}) = \check{x} - \langle \alpha, \check{x} \rangle \check{\alpha}$ preserve Φ and $\check{\Phi}$.

(The last axiom is equivalent to: The endomorphisms w_α preserve Φ and generate a finite group.)

Theorem 1.1.3 (Classification over the algebraic closure)**.** *Assume k algebraically closed. For any root datum there exists a connected reductive group G and a maximal torus $A \subset G$ over k which give rise to that root datum; the pair (G, A) is unique up to isomorphism.*

Some more definitions: Let $\mathcal{R} \subset X, \check{\mathcal{R}} \subset \check{X}$ be the lattices spanned by roots, resp. co-roots, $V = X \otimes \mathbb{R}$ and $\check{V} = \check{X} \otimes \mathbb{R}$. The root datum is called *semisimple* if $\mathcal{R} \otimes \mathbb{R} = V$; this is equivalent to the corresponding group being semisimple. Assume that this is the case. Let $\mathcal{P} \subset V$, $\check{\mathcal{P}} \subset \check{V}$ be the duals of $\check{\mathcal{R}}, \mathcal{R}$ respectively. Hence $\mathcal{R} \subset X \subset \mathcal{P}$ and $\check{\mathcal{R}} \subset \check{X} \subset \check{\mathcal{P}}$ We say that the root datum is *simply-connected* if $X = \mathcal{P}$ and *adjoint* if $X = \mathcal{R}$. This is known to be equivalent to the corresponding group being so. (Simply connected means that the étale fundamental group is trivial; in the complex case this is equivalent to the topological fundamental group being trivial; adjoint means that $G = G^{\mathrm{ad}}$.)

Examples: SL_n is simply connected, PGL_n is adjoint.

The elements of $\mathcal{P}$ (resp. $\check{\mathcal{P}}$) are called *weights* (resp. co-weights). Their significance is the following: Notice first that they depend only on the root system, not the root datum. Hence, they can be "seen" by the Lie algebra, which does not "see" the precise character lattice of the maximal torus. Over an algebraically closed field, dominant weights parametrize isomorphism classes of irreducible finite-dimensional modules for the Lie algebra. Then these representations can be "lifted" to the group (i.e. are the differential of a group representation) if and only if the corresponding weight is *integral*, which means that it belongs to X. For instance, if the group is simply connected then all representations can be lifted. (No surprise here!)

While it is difficult to describe morphisms between algebraic groups in general, it is easy to do so for morphisms which are *central isogenies*; that is, surjective morphisms with finite kernels belonging to the center. The combinatorial shadow of a central isogeny is the notion of *isogeny of root data*: Given two sets of root data $\Psi = (X, \Phi, \check{X}, \check{\Phi})$ and $\Psi' = (X', \Phi', \check{X}', \check{\Phi}')$ a homomorphism $X \to X'$ is called an isogeny if it is injective with image of finite index in X', it maps Φ bijectively to Φ' and its adjoing maps $\check{\Phi}'$ bijectively to $\check{\Phi}$.

Theorem 1.1.4. *Assume k algebraically closed. Consider root data as above and let (G, A), (G', A') be the corresponding groups and maximal tori according to 1.1.3. Every central isogeny $(G, A) \to (G', A')$ induces canonically an isogeny of root data $\Psi' \to \Psi$. Conversely, if $f : \Psi' \to \Psi$ is an isogeny, there exists a central isogeny $G \to G'$, unique up to automorphisms* $\mathrm{Inn}(a), a \in A$.

This theorem will allow us to understand automorphisms of reductive groups, which in turn will help us understand their forms over a non-algebraically closed field. First, we need to discuss based root data:

1.1.6.3 Weyl chambers and based root data

Given a root system (V, Φ) with Weyl group W, the complement $\mathring{V}$ of the union of fixed point hyperplanes of reflections in W is the union of a finite number of connected components, each of which is a fundamental domain for the action of W on $\mathring{V}$. Choosing such a *Weyl chamber* $\mathcal{C}$ gives rise to two translation-invariant orderings on V: The stronger of the two is the one defined by C, and elements which are greater or equal to zero with respect to this (i.e. elements of $\overline{\mathcal{C}}$) are called *dominant*. The weaker is the one defined by the cone spanned by the roots α such that $\langle c, \check{\alpha} \rangle > 0$ for every $c \in \mathcal{C}$. These roots are called *positive* and their set will be denoted by Φ^+. Notice that $\forall \alpha \in \Phi, \alpha \in \Phi^+$ or $-\alpha \in \Phi^+$. When expressing an inequality in V we will, by default, refer to this weaker ordering; when we say "an ordering of the roots" or "a choice of positive roots" we mean one induced by the choice of a Weyl chamber. The irreducible elements of Φ^+ (i.e. those which cannot be written as a sum of others) form the set of *simple roots*, usually denoted by Δ, and they are a basis for V. A choice of positive roots induces a choice of positive co-roots in the dual root system, such that $\alpha > 0 \iff \check{\alpha} > 0$.

If (G, A) is an algebraic group with a maximal torus (over an algebraically closed field), (V, Φ) is the corresponding root system and B is a Borel subgroup containing A, then it is easy to see that the roots of $\mathfrak{b}$ form the set of positive roots with respect to a choice of Weyl chamber. One can see that the set of Weyl chambers is in bijection with the Borel subgroups of G containing A. A root datum Ψ, together with a choice of positive roots as above, is called a *based root datum*. An automorphism of a based root datum is an automorphism of the root datum which preserves the set of positive roots.

1.1.6.4 Automorphisms

Now we study automorphisms of a reductive group G. Fix a maximal torus A, a Borel $B \supset A$ (corresponding to a choice of positive roots) and non-zero elements $u_\alpha \in \mathfrak{u}_\alpha$ for all $\alpha > 0$. Given an automorphism ϕ of G, by the fact that all Borels are conjugate we can compose it with an inner automorphism to get an automorphism ϕ' which preserves B. Moreover we can compose that with $\mathrm{Inn}(a)$, for some $a \in A$, to get an automorphism ϕ'' which fixes the u_α's. It is a fact that the Borel subgroup is its own normalizer; from this it can be seen that ϕ'' is the only element in its Inn-coset which preserves the data $(A, B, \{u_\alpha\}_\alpha)$.

Hence:

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) = \mathrm{Aut}(G, A, B, \{u_\alpha\}_\alpha) \to 1$$

and this sequence splits.

It is clear from our discussion of central isogenies that automorphisms of $(G, A, B, \{u_\alpha\}_\alpha)$ are in bijection with automorphisms of the corresponding based root datum. (For a semisimple simply connected or adjoint group the latter are, in turn, the same as *isomorphisms of the Dynkin diagram.*)

In the next lecture, we will use our knowledge of automorphisms to discuss the Galois action on a group and classify forms of the group over a non-algebraically closed field.

1.1.7 Parabolic subgroups

Theorem 1.1.5. *Let G be a reductive group and fix a based root datum corresponding to (A, B). Let Δ denote the set of simple positive roots. The parabolic subgroups of G containing B are in natural bijection with subsets of Δ. Let $I \subset \Delta$ and let P_I be the corresponding subgroup, U_I its unipotent radical. Then $P_I = L_I \ltimes U_I$ where L_I is the following subgroup: We have $\mathcal{Z}(L_I) = \{a \in A | \alpha(a) = 0$ for all $\alpha \in I$; and L_I is the centralizer in G of $\mathcal{Z}(L_I)$.*

Such a subgroup L_I is called a *Levi subgroup.* The equalities above also hold at the level of k-points for any field k, if P_I is defined over k: $P_I(k) = L_I(k)U_I(k)$. Moreover, $(G/P)(k) = G(k)/P(k)$ for every parabolic. Having fixed a Borel subgroup B, a parabolic which contains it is called a *standard parabolic.* Since all Borels are conjugate, all parabolics are conjugate to a (unique) standard one.