Proof of Infinitely Many Primes

by L. Shorser

The following proof is attributed to Eulclid (c. 300 b.c.).

Theorem: There are infinitely many prime numbers.

Proof. A prime number is a natural number with exactly two distinct divisors: 1 and itself. Let us assume that there are finitely many primes and label them $p_1, \ldots, p_n$. We will now construct the number P to be one more than the product of all finitely many primes:

$$P = p_1 p_2 \cdots p_n + 1.$$

The number P has remainder 1 when divided by any prime p_i, $i = 1, \ldots, n$, making it a prime number as long as $P \neq 1$.

Since 2 is a prime number, the list of p_i's is non-empty. It follows that P is greater than one and so has two distinct divisors. It is therefore a prime number.

It can also be seen from the definition of P that it is strictly greater than any of the p_i's. This contradicts our assumption that there are finitely many prime numbers. Therefore, there are infinitely many prime numbers. □

Alternatively, one can leave out the assumption and let $p_1, \ldots, n$ be any arbitrary finite list of prime numbers. Then the conclusion would state that for any finite list of prime numbers, it is possible to construct a larger prime than any on the list. This method uses induction.

<u>Structure:</u>

<u>Assumptions:</u>

<u>Definitions and Results:</u>

<u>Scope:</u>