Induction Variations

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<u>**Example 1:**</u> Prove that the probability of flipping all heads in any set of consecutive coin tosses is always greater than zero.

<u>Base Case:</u> The probability of flipping heads on a single coin toss is $\frac{1}{2}$, which is greater than zero.

<u>Induction Hypothesis:</u> Denote the probability of flipping all heads in a set of k consecutive coin tosses by $\mathrm{P}(k)$. We assume that $\mathrm{P}(k) > 0$.

<u>Case where $n = k+1$:</u> $\mathrm{P}(k+1) = \frac{1}{2}\mathrm{P}(k) > 0$ by induction hypothesis.

<u>Note:</u> This proof could also have been done directly (without induction) by calculating $\mathrm{P}(k) = \frac{1}{2^k}$ and noticing that it is positive for every finite value of k.

<u>**Example 2:**</u> Let $f_n(x) = nx + a$ be a family of functions $\forall\ n \in \mathbb{N}$ where a is a fixed positive real number. Prove that $f_n(x) \geq a$ for all $x \in [0, \infty)$.

<u>Base Case:</u> $f_1(x) = x + a$. When $x \in [0, \infty)$, we can say that $0 \leq x \leq \infty$.
Adding a to all sides, we get: $a \leq x + a \leq \infty$.
The left inequality can be written: $x + a \geq a$ which is the same as $f_1(x) \geq a$.

<u>Induction Hypothesis:</u> Assume that $f_k(x) \geq a$ on $x \in [0, \infty)$.

<u>Case where $n = k+1$:</u> $f_{k+1}(x) = (k+1)x + a = kx + x + a = x + (kx + a) = x + f_k(x)$. By the induction hypothesis, $f_k(x) \geq a$ on $x \in [0, \infty)$. Since x is positive on this interval, $f_k(x) + x > f_k x$. Therefore, $f_{k+1}(x)$ is even bigger than $f_k(x)$ which was already greater than or equal to a on this interval. In conclusion, we can now say that $f_{k+1}(x) \geq a$ on $x \in [0, \infty)$.

<u>Note:</u> This proof could also have been done directly using algebra, calculus, or functional comparisons as opposed to induction.