Glossary of Terms

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$\in$: This symbol means "is an element of". For example, "$x \in \mathbb{R}$" is the symbolic expression for the phrase "x is an element of the set of real numbers".

e.g.: This stands for "exempli gratia". In English, this means "for example".

i.e.: This stands for "id est". In English, this would generally be expressed by the words "that is" or "in other words".

Arbitrary: An *arbitrary* element of a set is an abstract element with properties that are shared with all elements in the set and no other properties. For example, let x be an arbitrary student at the University of Toronto. Then x is a student, is enrolled in at least one course at U of T, and is human. However, x cannot be considered to be male or female since neither of these properties is shared among **all** students at U of T.

Axiom: A statement which has not been proven, but is generally accepted as true is called an *axiom*. For example, the Axiom of Choice cannot be proven. However, many important results in mathematics rely on the idea that it is true.

Claim: see *Proposition*

Contradiction: A *contradiction* is a set of ideas which cannot both be true. For example, the statement "x is a positive and negative non-zero number" is a contradiction since no non-zero number can be both positive and negative.

Contrapositive: The *contrapositive* of an implication "$A \to B$" is "$\sim B \to \sim A$". In other words, it is the case where the implied idea being false implies the premise is also false.

Converse: The *converse* of the implication "$A \to B$" is "$B \to A$". In other words, it is the reverse implication.

Corollary: A *corollary* is a result that follows directly from a theorem. For example, Euclid's first theorem states that if a prime p divides evenly into the product bc, then p is a factor of at least one of b and/or c. The fact that all positive integers greater than 1 factor into a product of prime numbers is a corollary of this theorem.

Definition: An equivalence (if and only if statement) where one side of the double implication is a label. For example, "a figure is a triangle if and only if it is a polygon with three sides" is the definition of the label "triangle".

Equivalence ("iff", if and only if): This is a statement where implication goes in both directions. For example, "x is a rational number if and only if it can be expressed as a fraction" is an equivalence between the statements "x is a rational number" and "x can be expressed as a fraction".

If and only if, "iff": (see *equivalence*)

Implication: Any statement that has an if-then structure is called an *implication*. For example, the statements "If it rains, I will take an umbrella" can be symbolized by $A \to B$. The arrow represents the word "implies".

Lemma: A *lemma* is a statement which is proven so that it can be used in the proof of a theorem (or other large result). For example, before you can prove that all complex numbers can be expressed as 1 over another complex number, you must first prove the following lemma: The product of two complex numbers is also a complex number.

Proof By Contradiction: This proof structure starts with the assumption that a statement is false. Then the assumption is used to generate two contradictory statements. It can then be concluded that the original statement was actually true. For example, to prove by contradiction that $\sqrt{2}$ is irrational, we assume that this statement is false. In other words, we assume that $\sqrt{2}$ is rational and can therefore be expressed as a fraction ($\sqrt{2} = \frac{p}{q}$, where p and q have no common factors). Squaring both sides and multiplying them both by q^2 and dividing them both by 2, we get that $q^2 = \frac{p^2}{2}$. In other words, since q^2 is a whole number p^2 and therefore p must be divisible by 2. However, if p is divisible by 2, p^2 is divisible by 4. So q^2 (which is equal to half of p^2) is also divisible by 2. This contradicts the statement that $\sqrt{2}$ can be expressed as a fraction where the numerator and denominator have no factors in common.

Proof By Contrapositive: This proof structure replaces the proof of "$A \to B$" with the proof that "$\sim B \to \sim A$". For example, instead of proving that x being an integer implies that x is a real number, we can prove: If x is not a real number, it could not have been an integer.

Proof By Counterexample: This proof structure allows us to prove that a property is not true by providing an example where it does not hold. For example, to prove that "not all triangles are obtuse", we give the following counter example: the equilateral triangle having all angles equal to sixty. In this case, there are infinitely many counterexample. However, it only takes one.

Proposition: A *proposition* is a result that is being proven in the middle of a discussion. It is a conversational name for any result, even potentially a theorem if the theorem is being given out of context. Such a result can also be called a *Claim*.

Quantifiers: A *quantifier* is an expression that indicates scope. The main logical quantifiers are $\forall$ (for all) and $\exists$ (there exists). For example, the statement "$\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $x + y = 0$" means "for all integers x, there exists an integer y such that their sum is zero". The symbol $\exists!$ meanse "there exists a unique" (e.g., "$\exists! x \in \mathbb{Q} \ldots$" means "There exists a unique rational number x such that ...")

Theorem: A *theorem* is a major result that has been proven. For example, The Fundamental Theorem of Arithmetic states that every positive integer greater than 1 can be expressed as a product of prime numbers.

WOLOG: This stands for "without loss of generality". This phrase is used when a condition is being imposed that does not limit the scope of a proof. For example, one can say "Without loss of generality, we can let the y-intercept of this parabola be a real number". This statement does not eliminate the case where the parabola has no real y-intercept because all parabolas must intersect the y-axis somewhere.