Proof By Contradiction

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This proof structure starts with the assumption that a statement is false. Then the assumption is used to generate two contradictory statements. It can then be concluded that the original statement was actually true.

For example, to prove by contradiction that $\sqrt{2}$ is irrational, we assume that this statement is false. In other words, we assume that $\sqrt{2}$ is rational.

Claim. The square root of 2 is irrational.

Proof. Assume that $\sqrt{2}$ is rational and can therefore be expressed as a fraction.

$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers with no common factors and $q \neq 0$.. Squaring both sides we get

$$2 = \frac{p^2}{q^2}$$

Multiplying them both by q^2, we get

$$2q^2 = p^2$$

This statement implies that, since $2q^2$ is an even number, p^2 must be even as well. If the factors of p are p_1, p_2, ..., p_k, then the factors of p^2 are p_1^2, p_2^2, ..., p_k^2. So if p^2 has 2 as a factor, it must be divisible by 4. So the right side of the equation above is divisible by 4. This implies that q is also divisible by 2. This contradicts the statement that $\sqrt{2}$ can be expressed as a fraction where the numerator and denominator have no factors in common.

Therefore, $\sqrt{2}$ is irrational. □