Chapter 6

Definitions of curvature bounded below

In section 6.1, we will start with a global definition of Alexandrov spaces via (1+3)-point comparison. In section 6.2, we give a number of equivalent angle comparison definitions. These definitions give the easiest way to adapt your Euclidean intuition to Alexandrov's world. Later we give a few equivalent global definitions via concavity of distance functions and development of geodesics (section 6.3). There is yet another definition via extension of short maps on 4-point subsets (section 26.1). In chapter **??**, we also discuss Wald's original definition which gives a uniform approach to definitions of spaces with lower and upper curvature bound.

In section 6.4, we give local versions of each definition and prove the globalization theorem.

Further, we discuss some basic corollaries of the definition: first variation formula and non-splitting of geodesics in section 6.5 and some specific properties of spaces with positive curvature bound in section 6.6.

6.1 (1+3)-point comparison.

6.1.1. Definition. *A quadrouple of points p, x^1, x^2, x^3 in a metric space satisfies* (1+3)-point comparison *if*

$$\tilde\measuredangle^\kappa(p\,{}^{x^1}_{x^2}) + \tilde\measuredangle^\kappa(p\,{}^{x^2}_{x^3}) + \tilde\measuredangle^\kappa(p\,{}^{x^3}_{x^1}) \leqslant 2\cdot\pi. \qquad ❶$$

or at least one of the model angles $\tilde\measuredangle^\kappa(p\,{}^{x^i}_{x^j})$ is not defined.

6.1.2. Definition. *A complete intrinsic space $\mathcal{L}$ is called an Alexandrov space with curvature $\geqslant \kappa$ (briefly, $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$) if any quadruple $p, x^1, x^2, x^3 \in \mathcal{L}$ satisfies (1+3)-point comparison.*

Remarks

- ⋄ We say that $\mathcal{L}$ has curvature bounded below (briefly, $\mathcal{L}$ is a CBB-space or $\mathcal{L} \in \mathrm{CBB}$) if there is $\kappa \in \mathbb{R}$, such that $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$.

- ⋄ Alexandrov spaces with curvature bounded below will be mostly denoted by $\mathcal{L}$, for $\mathcal{L}$ower curvature bound.
- ⋄ If $\kappa > 0$, in the definition of spaces with curvature $\geqslant \kappa$, most authors assume in addition that $\operatorname{diam}\mathcal{L} \leqslant \varpi^\kappa$, or equivalently, that $\mathcal{L}$ is not isometric to one of the exceptional spaces, see 6.6.2. We don't assume that. In particular, we consider the real line to have curvature $\geqslant 1$.
- ⋄ If $\kappa < K$, the following inclusion holds:

$$\mathrm{CBB}\lfloor\kappa\rfloor \supset \mathrm{CBB}\lfloor K\rfloor,$$

but in the case $K > 0$, is not at all trivial. It will be shown only in 6.4.5.

It is straightforward to check the following claim (for definitions of ω-limit and ω-product, see section 5.3).

6.1.3. Proposition. *Let ω be a nonprinciple ultrafilter on $\mathbb{N}$ and $(\mathcal{L}_n, \star_n) \xrightarrow{\omega} (\mathcal{L}_\omega, \star_\omega)$ and $\kappa_n \xrightarrow{\omega} \kappa_\omega$. Then*

$$\mathcal{L}_n \in \mathrm{CBB}\lfloor\kappa_n\rfloor \quad\Longrightarrow\quad \mathcal{L}_\omega \in \mathrm{CBB}\lfloor\kappa_\omega\rfloor.$$

Moreover, for any complete metric space $\mathcal{L}$,

$$\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor \quad\Longleftrightarrow\quad \mathcal{L}^\omega \in \mathrm{CBB}\lfloor\kappa\rfloor,$$

where $\mathcal{L}^\omega$ denotes ultraproduct of $\mathcal{L}$.

General complete intrinsic spaces might have no geodesics (see exercise 1). For Alexandrov spaces the situation is different; all Alexandrov's spaces are G_δ-geodesic in the sense of the following definition:

6.1.4. Definition. *A complete intrinsic space $\mathcal{X}$ is called* G_δ-geodesic *if for any point $p \in \mathcal{X}$ there is a dense G_δ-set $W_p \subset \mathcal{X}$ such that for any $q \in W_p$ there is geodesic $[pq]$.*

6.1.5. Definition. *Let $\mathcal{X}$ be a metric space and $p \in \mathcal{X}$. Let us call $q \in \mathcal{X}$* p-straight *(briefly, $q \in \mathrm{Str}(p)$) if*

$$\varlimsup_{r\to q} \frac{|pr| - |pq|}{|qr|} = 1.$$

For an array of points $x^1, x^2, \dots, x^{\bar{k}}$, we will use short notation

$$\mathrm{Str}(x^1, x^2, \dots, x^{\bar{k}}) = \bigcap_{i=1}^{\bar{k}} \mathrm{Str}(x^i).$$

6.1.6. Plaut's theorem. *Let $\mathcal{L} \in \mathrm{CBB}$ and $p \in \mathcal{L}$. Then the set $\mathrm{Str}(p)$ is a dense G_δ-set, and for any $q \in \mathrm{Str}(p)$ there is a unique geodesic $[pq]$.*

In particular, $\mathcal{L}$ is G_δ-geodesic.

We repeat the proof from [Plaut 02, Th. 27]. First let us prove a lemma in metric geometry.

6.1.7. Lemma. *Let $\mathcal{X}$ be a complete intrinsic space. Then for any $p \in \mathcal{X}$ the set of* $\operatorname{Str}(p) \subset \mathcal{X}$ *is a dense G_δ-set.*

Proof. Set

$$\Omega_n = \left\{\, q \in \mathcal{X} \;\middle|\; (1 - \tfrac{1}{n})\cdot|qr| < |pr| - |pq| < \tfrac{1}{n} \text{ for some } r \in \mathcal{X} \,\right\}.$$

Clearly Ω_n is open; let us show that it is dense in $\mathcal{X}$. Assume contrary, then there is a point $x \in \mathcal{X}$ such that $\mathrm{B}(x, \varepsilon) \cap \Omega_n = \varnothing$ for some $\varepsilon > 0$. Since metric on $\mathcal{L}$ is intrinsic, for any $\delta > 0$, there exists a point $y \in \mathcal{L}$ such that $|xy| < \frac{\varepsilon}{2} + \delta$ and $|py| < |px| - \frac{\varepsilon}{2} + \delta$. For small enough ε and δ, that implies $(1 - \frac{1}{n})\cdot|yx| < |px| - |py| < \frac{1}{n}$, i.e. $y \in \Omega_n$, a contradiction.

Since $\operatorname{Str}(p) = \bigcap_{n\in\mathbb{N}} \Omega_n$, the result follows. □

Remark. An alternative proof can be built on the fact that geodesics do not split (6.5.1), and the following corollary of lemma 5.3.4: *Assume $\mathcal{X}$ is a complete intrinsic space and $p, q \in \mathcal{X}$ can not be joined by a geodesic in $\mathcal{X}$ then there is at least two distinct geodesics between p and q in the ultraproduct $\mathcal{X}^{\omega}$.*

Proof of theorem 6.1.6. Let $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$. We will show that if $q \in \operatorname{Str}(p)$, then there is unique minimizing geodesic connecting p and q. Thus theorem will follow from lemma 6.1.7.

Note that it is enough to show that for all sufficiently small $t > 0$ there is unique point z such that

$$t = |qz| = |pq| - |pz|. \qquad ❷$$

First let us show uniqueness. Assume z and z' both satisfy ❷. Take a sequence $r_n \to q$ so that $(|pr_n| - |pq|)/|qr_n| \to 1$. From triangle inequality,

$$|zr| - |zq|, \quad |z'r| - |z'q| \geqslant |pr| - |pq|;$$

thus, as $n \to \infty$,

$$\frac{|zr_n| - |zq|}{|qr_n|}, \quad \frac{|z'r_n| - |z'q|}{|qr_n|} \to 1.$$

Therefore $\tilde\measuredangle^\kappa(q\,{}^{z}_{r_n})$, $\tilde\measuredangle^\kappa(q\,{}^{z'}_{r_n}) \to \pi$. (Here we use that t is small, otherwise if $\kappa > 0$ the angles might be undefined.) From (1+3)-point comparison (6.1.2), $\tilde\measuredangle^\kappa(q\,{}^{z}_{z'}) = 0$ and thus $z = z'$.

The proof of existence is similar. Choose sequence r_n as above. Since $\mathcal{L}$ is complete intrinsic space, there is a sequence $z_\hbar \in \mathcal{L}$ such that $|qz_\hbar|$, $|pq| - |pz_\hbar| \to t$ as $\hbar \to \infty$. Then

$$\lim_{n\to\infty} \lim_{\hbar\to\infty} \tilde\measuredangle^\kappa(q\,{}^{z_\hbar}_{r_n}) = \pi.$$

Thus, for any $\varepsilon > 0$ and large enough $n, \hbar$ we have $\tilde\measuredangle^\kappa(q\,{}^{z_\hbar}_{r_n}) > \pi - \varepsilon$. From (1+3)-point comparison (6.1.2), for all large $\hbar$ and j, we have $\tilde\measuredangle^\kappa(q\,{}^{z_\hbar}_{z_j}) < 2\cdot\varepsilon$ and thus

$$|z_\hbar z_j| < \varepsilon\cdot\mathrm{const}(\kappa, t);$$

i.e. $\{z_n\}$ is a Cauchy sequence, and $z = \lim_n z_n$ satisfies ❷. □

6.2 More angle comparisons.

Now we are ready to give a number of equivalent definitions of Alexandrov space:

6.2.1. Theorem. *Let $\mathcal{L}$ be a complete intrinsic space. Then*

a) $\mathcal{L} \in \mathrm{CBB}[\kappa]$

if and only if $\mathcal{L}$ is G_δ-geodesic and one of the following conditions holds for all $p, x, y \in \mathcal{L}$, once $\tilde\triangle^\kappa(pxy)$ is defined:

b) (adjacent angle comparison) for any minimizing geodesic $[xy]$ and $z \in]xy[$, $z \neq p$ we have

$$\tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) \leqslant \pi.$$

c) (point-on-side comparison) for any minimizing geodesic $[xy]$ and $z \in]xy[$, we have

$$\tilde\measuredangle^\kappa(x\,{}^{p}_{y}) \leqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{z});$$

or, equivalently,

$$|\tilde p\tilde z| \leqslant |pz|,$$

where $[\tilde p\tilde x\tilde y] = \tilde\triangle^\kappa(pxy)$, $\tilde z \in]\tilde x\tilde y[$, $|\tilde x\tilde z| = |xz|$.

d) (hinge comparison)for any hinge $[x\,{}^{p}_{y}]$, the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and

$$\measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{y}).$$

or equivalently

$$\tilde\Upsilon^\kappa[x\,{}^{p}_{y}] \geqslant |py|.$$

Moreover, if $z \in]xy[$, $z \neq p$ then for any two hinges $[z\,{}^{p}_{y}]$ and $[z\,{}^{p}_{x}]$ with common side $[zp]$

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi.$$

Remarks. A bit weaker form of d is given in 6.4.6. See also open problem 6.7.1.

Proof. By theorem 6.1.6, we have that (a) imples that $\mathcal{L}$ is G_δ-geodesic.

(a) ⇒ (b). Since $z \in]xy[$, we have $\tilde\measuredangle^\kappa(z\,{}^{x}_{y}) = \pi$. Thus, (1+3)-point comparison

$$\tilde\measuredangle^\kappa(z\,{}^{x}_{y}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) \leqslant 2\cdot\pi$$

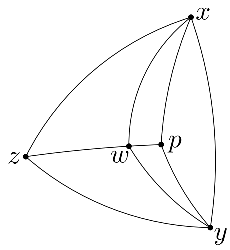
implies

$$\tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) \leqslant \pi.$$

(b) ⇔ (c). Follows directly from Alexandrov's lemma (3.2.1).

(b) + (c) ⇒ (d). From (c) we get that for $\bar p \in]xp]$ and $\bar y \in]xy]$ the function $(|x\bar p|, |x\bar y|) \mapsto \tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})$ is nonincreasing in each argument. In particular, $\measuredangle[x\,{}^{p}_{y}] = \sup\{\tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})\}$. Thus, $\measuredangle[x\,{}^{p}_{y}]$ is defined and it is at least $\tilde\measuredangle^\kappa(x\,{}^{p}_{y})$.

From above and (b), it follows that $\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi$.



(d) ⇒ (a). Assume first that $\mathcal{L}$ is geodesic. Consider point $w \in]pz[$ close to p. From (d), it follows that

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{x}_{p}] \leqslant \pi \quad\text{and}\quad \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{y}_{p}] \leqslant \pi.$$

Since $\measuredangle[w\,{}^{x}_{y}] \leqslant \measuredangle[w\,{}^{x}_{p}] + \measuredangle[w\,{}^{y}_{p}]$ (see 3.3.2), we get

$$\measuredangle[w\,{}^{x}_{z}] + \measuredangle[w\,{}^{y}_{z}] + \measuredangle[w\,{}^{x}_{y}] \leqslant 2{\cdot}\pi.$$

Applying the first inequality in (d), we obtain

$$\tilde\measuredangle^{\kappa}(w\,{}^{x}_{z}) + \tilde\measuredangle^{\kappa}(w\,{}^{y}_{z}) + \tilde\measuredangle^{\kappa}(w\,{}^{x}_{y}) \leqslant 2{\cdot}\pi.$$

Passing to the limits $w \to p$, we obtain

$$\tilde\measuredangle^{\kappa}(p\,{}^{x}_{z}) + \tilde\measuredangle^{\kappa}(p\,{}^{y}_{z}) + \tilde\measuredangle^{\kappa}(p\,{}^{x}_{y}) \leqslant 2{\cdot}\pi.$$

Finally, if $\mathcal{L}$ is not geodesic, we can apply above arguments for sequences of points $p_n, w_n \to p$, $p_n \in \mathrm{Str}(z)$, $w_n \in]zp_n[$ and $x_n \to x$, $y_n \to y$, $x_n, y_n \in \in \mathrm{Str}(w_n)$. □

Proof. See the first part of (b)+(c)⇒(d) in the proof of 6.2.1. □

The following corollary restates the part (b)+(c)⇒(d) in the proof of 6.2.1. The monotonicity of model angle with respect to adjacent sidelengths (7.2.2a) was named the *convexity property* by Alexandrov.

6.2.2. Corollary. *Let $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$, $p, x, y \in \mathcal{L}$, $\tilde\triangle^{\kappa}(pxy)$ is defined and there is a geodesic $[xy]$. Then for $\bar y \in]xy]$ the function*

$$|x\bar y| \mapsto \tilde\measuredangle^{\kappa}(x\,{}^{p}_{\bar y})$$

is nonincreasing.

In particular, if geodesic $[xp]$ exists and $\bar p \in]xp]$ then

a) the function

$$(|x\bar y|, |x\bar p|) \mapsto \tilde\measuredangle^{\kappa}(x\,{}^{\bar p}_{\bar y})$$

is nonincresing in each argument

b) The angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and

$$\measuredangle[x\,{}^{p}_{y}] = \sup\left\{\, \tilde\measuredangle^{\kappa}(x\,{}^{\bar p}_{\bar y}) \;\middle|\; \bar p \in]xp]\,,\ \bar y \in]xy] \,\right\}.$$

6.2.3. Corollary. *Let $[z\,{}^{x}_{p}]$ and $[z\,{}^{y}_{p}]$ be two hinges with comon side $[zp]$. Assume that points p, x, y and z are distinct and $z \in [xy]$, then*

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] = \pi.$$

Proof. From hinge comparison (6.2.1d) we have that both angles $\measuredangle[z\,{}^{p}_{y}]$ and $\measuredangle[z\,{}^{p}_{x}]$ are defined and

$$\measuredangle[z\,{}^{p}_{y}] + \measuredangle[z\,{}^{p}_{x}] \leqslant \pi.$$

Clearly $\measuredangle[z\,{}^{x}_{y}] = \pi$. Thus, the result follows from the triangle inequality for angles (3.3.2). □

6.3 Function comparison and development

In this section we will translate the angle comparison definitions (theorem 6.2.1) to a concavity-like property of the distance functions as defined in section 4.3. Conceptually, this is quite important step; we reformulate a global geometric condition into an infinitesimal condition on distance functions. We will give two formulation of this comparison: analytical, through function concavity, and geometrical, using development.

6.3.1. Theorem. *Let $\mathcal{L}$ be a complete intrinsic space. Then the following statements are equivalent:*

a) $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$.

b) (function comparison) $\mathcal{L}$ *is* G_δ*-geodesic and for any* $p \in \mathcal{L}$*, the function* $f = \mathrm{md}^\kappa \circ |p*|$ *satisfies the differential inequality*

$$f'' \leqslant 1 - \kappa{\cdot}f.$$

in $\mathrm{B}(p, \varpi^\kappa)$.

In particular, $\mathcal{L} \in \mathrm{CBB}\lfloor 0\rfloor$ if and only if for any $p \in \mathcal{L}$, the function $|p*|^2 \colon \mathcal{L} \to \mathbb{R}$ is 2-concave (see page 19).

Proof. Given a geodesic $[xy]$ in $\mathrm{B}(p, \varpi^\kappa)$, let $\ell = |xy|$. Consider model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}^\kappa(pxy)$ and set

$$\tilde{r}(t) = |\tilde{p}\gamma_{[\tilde{x}\tilde{y}]}(t)|, \qquad r(t) = |p\gamma_{[xy]}(t)|.$$

Clearly $\tilde{r}(0) = r(0)$ and $\tilde{r}(\ell) = r(\ell)$. Set $\tilde{f} = \mathrm{md}^\kappa \circ \tilde{r}$ and $f = \mathrm{md}^\kappa \circ r$. From 1.1.1*a* we get $\tilde{f}'' = 1 - \kappa{\cdot}\tilde{f}$.

Note that the point-on-side comparison (6.2.1*c*) for point p and geodesic $[xy]$ is equivalent to $\tilde{r} \leqslant r$. Since md^κ is increasing on $[0, \varpi^\kappa)$, $\tilde{r} \leqslant r$ is equivalent to $\tilde{f} \leqslant f$, which is Jensen's inequality (4.2.1*d*) for the function $t \mapsto \mathrm{md}^\kappa|p\gamma_{[xy]}(t)|$ on interval $[0, \ell]$. Hence the result. □

We will translate the above comparison into more geometric language, using the following definition of κ-development, introduced in [Alexandrov 57]. The definition is somewhat lengthy, but it defines a useful comparison object for a curve. It is usually easier to write proofs in terms of function comparison, but geometrically it is often easier to think in terms of κ-developments.

6.3.2. Lemma. *Let $\kappa \in \mathbb{R}$, $\mathcal{X}$ be a metric space, $\mathbb{I}$ be a real interval, $\gamma \colon \mathbb{I} \to \mathcal{X}$ be a 1-Lipschitz curve, $p \in \mathcal{X}$, $\tilde{p} \in \mathbb{M}^2[\kappa]$ and $0 < |p\gamma(t)| < \varpi^\kappa$ for all $t \in \mathbb{I}$. Then there exists a unique up to rotation curve $\tilde{\gamma} \colon \mathbb{I} \to \mathbb{M}^2[\kappa]$, parameterized by the arc-length such that $|\tilde{p}\tilde{\gamma}(t)| = |p\gamma(t)|$ for all t and the direction of $[\tilde{p}\tilde{\gamma}(t)]$ monotonically turns conterclockwise as t increases.*

6.3.3. Definition of development. *If $p, \tilde{p}, \gamma, \tilde{\gamma}$ be as in lemma 6.3.2 then $\tilde{\gamma}$ is called the* κ-development of γ with respect to p*; the point $\tilde{p}$ is called* the basepoint of development*. When we say that the κ-development of γ with respect to p is* defined *we allways assume that $0 < |p\gamma(t)| < \varpi^\kappa$ for all $t \in \mathbb{I}$.*

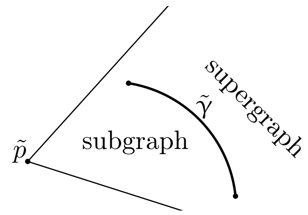
Proof of lemma 6.3.2. Consider functions $\rho, \vartheta \colon \mathbb{I} \to \mathbb{R}$ defined as

$$\rho(t) = |p\gamma(t)|, \qquad \vartheta(t) = \int_{t_0}^{t} \frac{\sqrt{1 - (\rho'(\tau))^2}}{\mathrm{sn}^\kappa \circ \rho(\tau)}{\cdot}d\tau,$$

where $t_0 \in \mathbb{I}$ is a fixed number and $\int$ stays for Lebesgue integral. Since γ is 1-Lipshitz, so is $\rho(t)$ and thus, the function ϑ is defined and non decreasing.

It is straightforward to check that (ρ, ϑ) describe $\tilde{\gamma}$ in polar coordinates on $\mathbb{M}^2[\kappa]$ with center at $\tilde{p}$. □

We need the following analogues of sub- and super-graphs and convex/concave functions, adapted to polar coordiantes in $\mathbb{M}^2[\kappa]$.



6.3.4. Definition. *Let $\tilde{\gamma} \colon \mathbb{I} \to \mathbb{M}^2[\kappa]$ be a curve and $\tilde{p} \in \mathbb{M}^2[\kappa]$ such that there is unique geodesic $[\tilde{p}\,\tilde{\gamma}(t)]$ for any $t \in \mathbb{I}$ and the direction of $[\tilde{p}\,\tilde{\gamma}(t)]$ turns monotonically as t grows.*

The set formed by all geodesics from $\tilde{p}$ to the points on $\tilde{\gamma}$ is called the subgraph *of $\tilde{\gamma}$ with respect to $\tilde{p}$.*

The set of all points $\tilde{x} \in \mathbb{M}^2[\kappa]$ such that a geodesic $[\tilde{p}\tilde{x}]$ intersect $\tilde{\gamma}$ is called the supergraph *of $\tilde{\gamma}$ with respect to $\tilde{p}$.*

The curve $\tilde{\gamma}$ is called convex (concave) with respect to $\tilde{p}$ *if the subgraph (supergraph) of $\tilde{\gamma}$ with respect to $\tilde{p}$ is convex.*

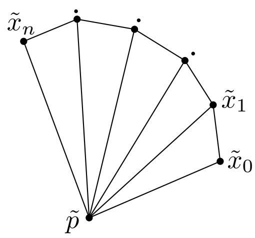
The curve $\tilde{\gamma}$ is called locally convex (concave) with respect to $\tilde{p}$ *if for any interior value t_0 in $\mathbb{I}$ there is a subsegment $(a,b) \subset \mathbb{I}$, $(a,b) \ni t_0$, such that the restriction $\tilde{\gamma}|_{(a,b)}$ is convex (concave) with respect to $\tilde{p}$.*

Note that if $\kappa > 0$ then the supergraph of a curve is a subgraph with respect to the opposite point.

For developments, all the notions above will be considered with respect to their basepoints.

In case $\tilde{\gamma}$ is a development, we will say it is *(locally) convex* if it is (locally) convex with respect to its basepoint.

6.3.5. Development comparison. $\mathcal{L} \in \mathrm{CBB}[\kappa]$ *if and only if $\mathcal{L}$ is G_δ-geodesic and for any point $p \in \mathcal{L}$ and any geodesic γ in $\mathrm{B}(p, \varpi^\kappa)\backslash\{p\}$, its κ-development with respect to p is convex.*



Proof; (⇒). By theorem 6.1.6, $\mathcal{L}$ is G_δ-geodesic.

Let $\gamma \colon [0,T] \to \mathrm{B}(p, \varpi^\kappa)\backslash\{p\}$ be a minimizing geodesic in $\mathcal{L}$. Consider partition $0 = t_0 < t_1 < \ldots < t_n = T$. Set $x_i = \gamma(t_i)$. Let us construct a chain of model triangles $[\tilde{p}\tilde{x}_{i-1}\tilde{x}_i] = \tilde{\triangle}^\kappa(p x_{i-1} x_i)$ in such a way that direction $[\tilde{p}\tilde{x}_i]$ turns conterclockwise as i grows. The condition 6.2.1*b* implies that

$$\measuredangle[\tilde{x}_i\,{}^{\tilde{x}_{i-1}}_{\tilde{p}}] + \measuredangle[\tilde{x}_i\,{}^{\tilde{x}_{i+1}}_{\tilde{p}}] \leqslant \pi. \qquad ❶$$

Further, since γ is a minimizing geodesic, we have that

$$\sum_{i=1}^{n} |x_{i-1}x_i| \leqslant |px_0| + |px_n|. \qquad ❷$$

If $\kappa \leqslant 0$ then ❷ immediately implies that

$$\sum_{i=1}^{n} \measuredangle[\tilde{p}\,{}^{\tilde{x}_i}_{\tilde{x}_{i-1}}] \leqslant \pi. \qquad ❸$$

If $\kappa > 0$ then ❸ follows from inequalities ❶ and ❷ by repeated application of Alexandrov's lemma 3.2.1.

Inequalities ❶ and ❸ imply that the polygon $[\tilde p\tilde x_0\tilde x_1\dots\tilde x_n]$ is convex.

Let us take finer and finer partitions and pass to the limit of the polygon $\tilde p\tilde x_0\tilde x_1\dots\tilde x_n$. We obtain a convex curvelinear triangle formed by a curve $\tilde\gamma\colon[0,T]\to\mathbb{M}^2[\kappa]$ which is the limit of broken line $\tilde x_0\tilde x_1\dots\tilde x_n$ and two geodesics $[\tilde p\,\tilde\gamma(0)]$, $[\tilde p\,\tilde\gamma(T)]$. Since $[\tilde p\tilde x_0\tilde x_1\dots\tilde x_n]$ is convex, the natural parametrization of $\tilde x_0\tilde x_1\dots\tilde x_n$ converges to a natural parametrization of $\tilde\gamma$ (see 1.2.1). Thus $\tilde\gamma$ is the κ-development of γ with respect to p.

($\Leftarrow$). Assuming convexity of the development, we will prove the point-on-side comparison (6.2.1c). We can assume that $p\notin[xy]$; otherwise the statement is trivial.

Set $T=|xy|$ and $\gamma(t)=\gamma_{[xy]}(t)$ it is a geodesic in $\mathrm{B}(p,\varpi^\kappa)\backslash\{p\}$. Let $\tilde\gamma\colon[0,T]\to\mathbb{M}^2[\kappa]$ be κ-development with base $\tilde p$ of γ with respect to p. Take a partition $0=t_0<t_1<\dots<t_n=T$ and set

$$\tilde y_i=\tilde\gamma(t_i)\quad\text{and}\quad\tau_i=|\tilde y_0\tilde y_1|+|\tilde y_1\tilde y_2|+\dots+|\tilde y_{i-1}\tilde y_i|.$$

Since $\tilde\gamma$ is convex, for a fine partition we have that broken line $\tilde y_0\tilde y_1\dots\tilde y_n$ is also convex. Applying Alexandrov's lemma (3.2.1) inductively to pairs of triangles $\tilde\triangle^\kappa\{\tau_{i-1},|\tilde p\tilde y_0|,|\tilde p\tilde y_{i-1}|\}$ and $\tilde\triangle^\kappa\{|\tilde y_{i-1}\tilde y_i|,|\tilde p\tilde y_{i-1}|,|\tilde p\tilde y_i|\}$, we obtain that sequence $\tilde\measuredangle^\kappa\{|\tilde p\tilde y_i|;|\tilde p\tilde y_0|,\tau_i\}$ is non increasing.

For finer and finer partitions we have

$$\max_i\{|\tau_i-t_i|\}\to 0.$$

Thus, point-on-side comparison (6.2.1c) follows. □

6.4 Local definitions and globalization

The aim of this section is to give local analogs of the definitions given above and to prove the equivalence to the global definitions — the globalization theorem.

First we need a local analog of the notion of a G_δ-geodesic space (definition 6.1.4):

6.4.1. Definition. *Let $\mathcal{X}$ be a complete intrinsic space. An open set $\Omega\subset\mathcal{X}$ is called G_δ-geodesic if for every point $p\in\Omega$ there is a G_δ-set W_p which is dense in Ω such that for any $q\in W_p$ there is a geodesic $[pq]$ in $\mathcal{X}$.*

6.4.2. Theorem. *Let $\mathcal{X}$ be a complete intrinsic space and $p\in\mathcal{X}$. Then the following conditions are equivalent:*

1) *(local (1+3)-point comparison) there is $R_1>0$ such that comparison*

$$\tilde\measuredangle^\kappa(q\,{}^{x^1}_{x^2})+\tilde\measuredangle^\kappa(q\,{}^{x^2}_{x^3})+\tilde\measuredangle^\kappa(q\,{}^{x^3}_{x^1})\leqslant 2\cdot\pi$$

holds for any $q,x^1,x^2,x^3\in\mathrm{B}(p,R_1)$.

2) *(local Kirszbraun property) there is $R_2>0$, such that for any 3-point subset F_3 and any 4-point subset $F_4\supset F_3$ in $\mathrm{B}(p,R_2)$, any short map $f\colon F_3\to\mathbb{M}^2[\kappa]$ can be extended as a short map $\bar f\colon F_4\to\mathbb{M}^2[\kappa]$ (so $f=\bar f|_{F_3}$).*

3) *(local function comparison) there is* $R_3 > 0$*, such that* $\mathrm{B}(p, R_3)$ *is* G_δ*-geodesic and for any* $q \in \mathrm{B}(p, R_3)$*, the function* $f = \mathrm{md}^\kappa \circ |q*|$ *satisfies* $f'' \leqslant 1 - \kappa{\cdot}f$ *in* $\mathrm{B}(p, R_3)$*.*
4) *(local adjacent angle comparison) there is* $R_4 > 0$ *such that* $\mathrm{B}(p, R_4)$ *is* G_δ*-geodesic and if* q *and geodesic* $[xy]$ *lie in* $\mathrm{B}(p, R_4)$ *and* $z \in]xy[$ *then*

$$\tilde\measuredangle^\kappa(z\,{}^q_x) + \tilde\measuredangle^\kappa(z\,{}^q_y) \leqslant \pi.$$

5) *(local point-on-side comparison) there is* $R_5 > 0$*, such that* $\mathrm{B}(p, R_5)$ *is* G_δ*-geodesic and if* q *and geodesic* $[xy]$ *lie in* $\mathrm{B}(p, R_5)$ *and* $z \in]xy[$*, we have*

$$\tilde\measuredangle^\kappa(x\,{}^q_y) \leqslant \tilde\measuredangle^\kappa(x\,{}^q_z);$$

or, equivalently,

$$|\tilde p\tilde z| \leqslant |pz|,$$

where $[\tilde p\tilde x\tilde y] = \tilde\triangle^\kappa(pxy)$*,* $\tilde z \in]\tilde x\tilde y[$*,* $|\tilde x\tilde z| = |xz|$*.*
6) *(local hinge comparison) there is* $R_6 > 0$*, such that* $\mathrm{B}(p, R_6)$ *is* G_δ*-geodesic and if* $x \in \mathrm{B}(p, R_6)$ *then (1) for any hinge* $[x\,{}^q_y]$*, the angle* $\measuredangle[x\,{}^q_y]$ *is defined and (2) if* $x \in]yz[$ *then*[1]

$$\measuredangle[x\,{}^q_y] + \measuredangle[x\,{}^q_z] \leqslant \pi.$$

Moreover, if hinge $[x\,{}^q_y]$ *lies in* $\mathrm{B}(p, R_6)$ *then*

$$\measuredangle[x\,{}^q_y] \geqslant \tilde\measuredangle^\kappa(x\,{}^q_y).$$

or equivalently

$$\tilde\Upsilon^\kappa[x\,{}^q_y] \geqslant |qy|.$$

7) *(local development comparison) there is* $R_7 > 0$*, such that* $\mathrm{B}(p, R_7)$ *is* G_δ*-geodesic and if a geodesic* γ *lies in* $\mathrm{B}(p, R_7)$ *and* $q \in \mathrm{B}(p, R_7)\backslash\gamma$ *then the* κ*-development* $\tilde\gamma$ *with respect to* q *is convex.*

Moreover, for each pair $i, j \in \{1, 2, \dots, 7\}$ *we can assume that* $R_i > \frac{1}{9}{\cdot}R_j$*.*

Proof. The proof of each equivalence repeats the proof of corresponding global equivalence in localized form; see proofs of theorems 6.2.1, 6.3.1, 6.3.3, 26.1.1. □

Now, let us give a local definition of lower curvature bound (6.1.2).

6.4.3. Definition. *Let* $\mathcal{X}$ *be a complete intrinsic space and* $p \in \mathcal{X}$*, we say that* $\mathcal{X}$ has curvature $\geqslant K$ at p *(briefly,* $\mathrm{curv}_p\,\mathcal{X} \geqslant K$*) if* p *satisfies any of the equivalent condition of theorem 6.4.2 for any* $\kappa < K$*.*

6.4.4. Globalization theorem. *Let* $\mathcal{L}$ *be a complete intrinsic metric space such that* $\mathrm{curv}_p\,\mathcal{L} \geqslant K$ *for any point* $p \in \mathcal{L}$ *then* $\mathcal{L} \in \mathrm{CBB}\lfloor K\rfloor$*.*

In the two-dimensional case this theorem was proved in [Alexandrov 57], later, in [Toponogov] it was proved for Riemannian manifolds of all dimensions. In the above generality, the theorem first appears in [BGP]; simplifications and

[1] Let us remind that $[x\,{}^q_y]$ and $[x\,{}^q_z]$ are short notations for pairs $([xq], [xy])$ and $([xq], [xz])$, thus these two hinges automatically have common side $[xq]$.

modifications were given in [Plaut 96], [Shiohama], [BBI 01]. Our proof, is based on presentations in [Plaut 96] and [BBI 01].

Applying this theorem, we are finally able to show that the expression "space with curvature $\geqslant \kappa$" makes sense:

6.4.5. Corollary. *$\mathcal{L} \in \mathrm{CBB}\lfloor K \rfloor$ if and only if $\mathcal{L} \in \mathrm{CBB}\lfloor \kappa \rfloor$ for any $\kappa < K$.*

Proof. Note that if $K \leqslant 0$, this statement follows directly from definition of Alexandrov space (6.1.2) and monotonicity of the function $\kappa \mapsto \tilde\measuredangle^{\kappa}(x\,{}^{y}_{z})$ (1.1.1 c).

The "if"-part also follows directly from definition.

For $K > 0$, the angle $\tilde\measuredangle^{K}(x\,{}^{y}_{z})$ might be undefined, but, it is defined for sufficiently small triangles. Thus, if $\kappa < K$ then

$$\mathcal{L} \in \mathrm{CBB}\lfloor K \rfloor \quad \Longrightarrow \quad \mathrm{curv}_p \mathcal{L} \geqslant \kappa \ \text{ for any } \ p \in \mathcal{L}.$$

Applying globalization theorem (6.4.4), we get the "only if"-part. ☐

For the proof of globalization theorem, will need three lemmas. First we need the following characterization of Alexandrov space; it coinsides with 6.2.1 d if $\kappa \leqslant 0$ and slightly weaker in case $\kappa > 0$.

6.4.6. Short hinge lemma. *Let $\mathcal{L}$ be a complete G_δ-geodesic space such that for any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$ the angle $\measuredangle[x\,{}^{p}_{y}]$ is defined and moreover, if $x \in \,]yz[$ then*

$$\measuredangle[x\,{}^{p}_{y}] + \measuredangle[x\,{}^{p}_{z}] \leqslant \pi.$$

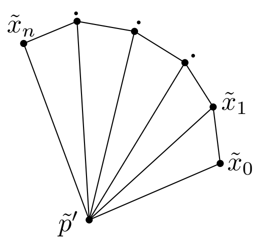
Assume that for any hinge $[x\,{}^{p}_{y}]$ in $\mathcal{L}$ we have

$$|px| + |xy| < \varpi^{\kappa} \quad \Rightarrow \quad \measuredangle[x\,{}^{p}_{y}] \geqslant \tilde\measuredangle^{\kappa}(x\,{}^{p}_{y}),$$

then $\mathcal{L} \in \mathrm{CBB}\lfloor \kappa \rfloor$.

Proof. If $\kappa \leqslant 0$, the lemma follows directly from the hinge comparison (6.2.1d). For $\kappa > 0$ the proof is very similar to the proof of equivalence of development comparison definition (6.3.5):

According to 6.3.5, it is enough to show that for any point $p \in \mathcal{L}$ and any geodesic $\gamma\colon [0,T] \to \mathrm{B}(p, \varpi^{\kappa}) \backslash \{p\}$, its κ-development with respect to p is convex.



Clearly $|p\gamma(t)| < \varpi^{\kappa} - \varepsilon$ for some $\varepsilon > 0$. Consider partition $0 = t_0 < t_1 < \ldots < t_n = T$ such that $|t_i - t_{i-1}| < \varepsilon$. Set $x_i = \gamma(t_i)$ and take $p' \approx p$, $p' \in \mathrm{Str}(x_0, x_1, \ldots, x_n)$. Construct a chain of model triangles $[\tilde{p}'\tilde{x}_{i-1}\tilde{x}_i] = \tilde\triangle^{\kappa}(p' x_{i-1} x_i)$ on such a way that direction $[\tilde{p}'\tilde{x}_i]$ turns conterclockwise as i grows. From the assumption of lemma,

$$\begin{aligned}\measuredangle[\tilde{x}_i\,{}^{\tilde{x}_{i-1}}_{\tilde{p}'}] + \measuredangle[\tilde{x}_i\,{}^{\tilde{x}_{i+1}}_{\tilde{p}'}] = \tilde\measuredangle^{\kappa}(x_i\,{}^{x_{i-1}}_{p'}) + \tilde\measuredangle^{\kappa}(x_i\,{}^{x_{i+1}}_{p'}) \leqslant \\ \leqslant \measuredangle[x_i\,{}^{x_{i-1}}_{p'}] + \measuredangle[x_i\,{}^{x_{i+1}}_{p'}] \ \leqslant \pi.\end{aligned}$$

Since γ is length minimizing, arguing exactly as in the proof of 6.3.5 we get that $\sum_{i=1}^{n} \measuredangle[\tilde{p}'\,{}^{\tilde{x}_i}_{\tilde{x}_{i-1}}] \leqslant \pi$ and the polygon $\tilde{p}'\tilde{x}_0\tilde{x}_1 \ldots \tilde{x}_n$ is convex.

Taking finer and finer partitions and passing to the limit $p' \to p$, the broken line $\tilde{x}_0\tilde{x}_1 \ldots \tilde{x}_n$ approaches development of γ with respect to p and the statement follows. ☐

The following lemma plays the key role in the proof of the globalization theorem

6.4.7. Key lemma. *Let $\kappa \in \mathbb{R}$, $0 < \ell \leqslant \varpi^\kappa$, $\mathcal{X}$ be a complete geodesic space and $p \in \mathcal{X}$ be a point such that $\operatorname{curv}_x \mathcal{X} \geqslant \kappa$ for any $x \in \mathrm{B}(p, 2{\cdot}\ell)$.*

Assume that for any point $q \in \mathrm{B}(p,\ell)$, comparison

$$\measuredangle[x\,{}^{y}_{q}] \geqslant \tilde\measuredangle^\kappa(x\,{}^{y}_{q})$$

holds for any hinge $[x\,{}^{y}_{q}]$ with $|xy| + |xq| < \tfrac{2}{3}{\cdot}\ell$. Then comparison

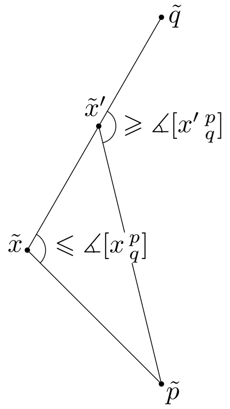
$$\measuredangle[x\,{}^{p}_{q}] \geqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{q})$$

holds for any hinge $[x\,{}^{p}_{q}]$ with $|xp| + |xq| < \ell$.

Proof of key lemma (6.4.7) . We will prove an equivalent inequality:

$$\tilde\Upsilon^\kappa[x\,{}^{p}_{q}] \geqslant |pq|. \qquad ❶$$

for any hinge $[x\,{}^{p}_{q}]$ with $|xp| + |xq| < \ell$.



Fix q. Given a hinge $[x\,{}^{p}_{q}]$ such that $\tfrac{2}{3}{\cdot}\ell \leqslant |px| + |xq| < \ell$, let us construct a new hinge $[x'\,{}^{p}_{q}]$ which is smaller, i.e.

$$|px| + |xq| \geqslant |px'| + |x'q| \qquad ❷$$

and such that

$$\tilde\Upsilon^\kappa[x\,{}^{p}_{q}] \geqslant \tilde\Upsilon^\kappa[x'\,{}^{p}_{q}]. \qquad ❸$$

Assume $|xq| \geqslant |xp|$, otherwise switch the roles of p and q in the following construction. Take $x' \in [xq]$ such that

$$|px| + 3{\cdot}|xx'| = \tfrac{2}{3}{\cdot}\ell \qquad ❹$$

Choose a geodesic $[x'p]$ and consider the hinge $[x'\,{}^{p}_{q}]$ fromed by $[x'p]$ and $[x'q] \subset [xq]$. (In fact the same argument as in 6.5.1 shows that condition $[x'q] \subset [xq]$ always holds.)

Then ❷ follows from the triangle inequality.

Further, note that we have $x, x' \in \mathrm{B}(p,\ell) \cap \mathrm{B}(q,\ell)$ and moreover $|px| + |xx'|, |px'| + |x'x| < \tfrac{2}{3}{\cdot}\ell$. In particular,

$$\measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{x'}) \ \text{ and } \ \measuredangle[x'\,{}^{p}_{x}] \geqslant \tilde\measuredangle^\kappa(x'\,{}^{p}_{x}). \qquad ❺$$

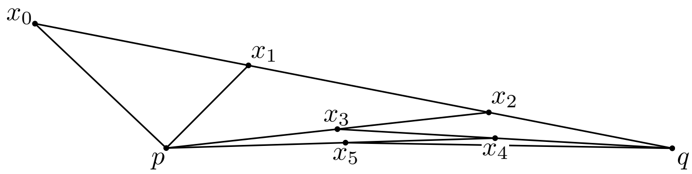
Now, let $[\tilde x\tilde x'\tilde p] = \tilde\triangle^\kappa(xx'p)$. Take $\tilde q$ on the extension of $[\tilde x\tilde x']$ beyond x' such that $|\tilde x\tilde q| = |xq|$ (and therefore $|\tilde x'\tilde q| = |x'q|$). From ❺,

$$\measuredangle[x\,{}^{p}_{q}] = \measuredangle[x\,{}^{p}_{x'}] \geqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{x'}) \quad\Rightarrow\quad \tilde\Upsilon^\kappa[x\,{}^{q}_{p}] \geqslant |\tilde p\tilde q|.$$

From the assumptions of the lemma, we have $\measuredangle[x'\,{}^{p}_{x}] + \measuredangle[x'\,{}^{p}_{q}] \leqslant \pi$; thus ❺ implies

$$\pi - \tilde\measuredangle^\kappa(x'\,{}^{p}_{x}) \geqslant \pi - \measuredangle[x'\,{}^{p}_{x}] \geqslant \measuredangle[x'\,{}^{p}_{q}].$$

Therefore $|\tilde p\tilde q| \geqslant \tilde\Upsilon^\kappa[x'\,{}^{q}_{p}]$ and ❸ follows.



Let us continue the proof. Set $x_0 = x$. Let us apply inductively the above construction to get a sequence of hinges $[x_n\,{}^{p}_{q}]$ with $x_{n+1} = x_n'$.

The sequence might terminate at some n only if $|px_n| + |x_nq| < \tfrac{2}{3}\cdot\ell$. In this case, by the assumptions of the lemma, $\tilde{\Upsilon}^\kappa[x_n\,{}^{p}_{q}] \geqslant |pq|$. From ❸, we get that the sequence $s_n = \tilde{\Upsilon}^\kappa[x_n\,{}^{p}_{q}]$ is decreasing. Hence inequality ❶ follows.

Otherwise, the sequence $r_n = |px_n| + |x_nq|$ is non-increasing and $r_n \geqslant \tfrac{2}{3}\cdot\ell$ for all n. Hence $r_n \to r$ as $n \to \infty$. Pass to a subsequence $y_{\bar{k}} = x_{n_{\bar{k}}}$ such that $y_{\bar{k}}' = x_{n_{\bar{k}}+1} \in [y_{\bar{k}}q]$. (In particular, $|y_{\bar{k}}q| \geqslant |y_{\bar{k}}p|$.) Clearly,

$$|py_{\bar{k}}| + |y_{\bar{k}}y_{\bar{k}}'| - |py_{\bar{k}}'| = r_{\bar{k}} - r_{\bar{k}+1} \underset{\bar{k}\to\infty}{\longrightarrow} 0. \qquad ❻$$

From ❹, it follows that $|y_{\bar{k}}y_{\bar{k}}'| > \tfrac{\ell}{100}$. From ❹ and ❻, it follows that $|py_{\bar{k}}| > \tfrac{\ell}{100}$ for all large $\bar{k}$. Therefore, $\tilde{\measuredangle}^\kappa(y_{\bar{k}}\,{}^{p}_{y_{\bar{k}}'}) \to \pi$. Since $\measuredangle[y_{\bar{k}}\,{}^{p}_{y_{\bar{k}}'}] \geqslant \tilde{\measuredangle}^\kappa(y_{\bar{k}}\,{}^{p}_{y_{\bar{k}}'})$, we have $\measuredangle[y_{\bar{k}}\,{}^{p}_{q}] = \measuredangle[y_{\bar{k}}\,{}^{p}_{y_{\bar{k}}'}] \to \pi$. Therefore,

$$|py_{\bar{k}}| + |y_{\bar{k}}q| - \tilde{\Upsilon}^\kappa[y_{\bar{k}}\,{}^{p}_{q}] \to 0.$$

(Here we used that $\ell \leqslant \varpi^\kappa$.) Together with the triangle inequality

$$|py_{\bar{k}}| + |y_{\bar{k}}q| \geqslant |pq|$$

this yields

$$\lim_{n\to\infty} \tilde{\Upsilon}^\kappa[y_{\bar{k}}\,{}^{p}_{q}] \geqslant |pq|.$$

Applying monotonicity of sequence $s_n = \tilde{\Upsilon}^\kappa[y_{\bar{k}}\,{}^{p}_{q}]$ we obtain ❶. □

We will also need the following general result on almost minima of functions on a metric space.

6.4.8. Lemma on almost minimum. *Let $\mathcal{X}$ be a complete space, $p \in \mathcal{X}$ and $r\colon \mathcal{X} \to \mathbb{R}$ be a function. Assume that for some $\varepsilon > 0$, the function r is strictly positive in $\overline{\mathrm{B}}(p, \tfrac{1}{\varepsilon^2}\cdot r(p))$ and $\underline{\lim}_n\, r(x_n) > 0$ for any convergent sequence $x_n \to x \in \overline{\mathrm{B}}(p, \tfrac{1}{\varepsilon^2}\cdot r(p))$.*

Then, there is a point $p^* \in \overline{\mathrm{B}}(p, \tfrac{1}{\varepsilon^2}\cdot r(p))$ such that

a) $r(p^*) \leqslant r(p)$ and

b) $r(x) > (1-\varepsilon)\cdot r(p^*)$ for any $x \in \overline{\mathrm{B}}(p^*, \tfrac{1}{\varepsilon}\cdot r(p^*))$.

Proof. Assume the statement is wrong. Then for any $x \in \mathrm{B}(p, \tfrac{1}{\varepsilon^2}\cdot r(p))$ with $r(x) \leqslant r(p)$, there is a point $x' \in \mathcal{X}$ such that

$$|xx'| < \tfrac{1}{\varepsilon}\cdot r(x) \quad\text{and}\quad r(x') \leqslant (1-\varepsilon)\cdot r(x).$$

Take $x_0 = p$ and consider a sequence of points (x_n) such that $x_{n+1} = x_n'$. Clearly

$$|x_{n+1}x_n| \leqslant \tfrac{r(p)}{\varepsilon}\cdot(1-\varepsilon)^n \quad\text{and}\quad r(x_n) \leqslant r(p)\cdot(1-\varepsilon)^n.$$

In particular, $|px_n| < \frac{1}{\varepsilon^2}\cdot r(p)$. Therefore (x_n) is a Cauchy sequence, $x_n \to x \in$ $\in \overline{\mathrm{B}}(p, \frac{1}{\varepsilon^2}\cdot r(p))$ and $\lim_n r(x_n) = 0$, a contradiction. □

Proof of the globalization theorem. Note that since for fixed points $x, y, z \in \mathcal{L}$, the function $\kappa \mapsto \tilde\measuredangle^\kappa(x\,{}^y_z)$ is continuous, it is enough to show that $\mathcal{L} \in \mathrm{CBB}[\kappa]$ for any $\kappa < K$. For the rest of the proof we fix some $\kappa < K$.

Exactly the same argument as in the proof of theorem 6.1.6 shows that $\mathcal{L}$ is G_δ-geodesic. By theorem 6.4.2-6, for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$ the angle $\measuredangle[x\,{}^p_y]$ is defined and moreover, if $x \in]yz[$ then

$$\measuredangle[x\,{}^p_y] + \measuredangle[x\,{}^p_z] \leqslant \pi.$$

Let us denote by $\mathrm{ComRad}(p, \mathcal{L})$ (which stands for *comparison radius* of $\mathcal{L}$ at p) the maximal value (possibly ∞) such that the comparison

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle^\kappa(x\,{}^p_y)$$

holds for any hinge $[x\,{}^p_y]$ with $|px| + |xy| < \mathrm{ComRad}(p, \mathcal{L})$.

As follows from 6.4.2-3, $\mathrm{ComRad}(p, \mathcal{L}) > 0$ for any $p \in \mathcal{L}$ and $\underline{\lim}\,\mathrm{ComRad}(p_n, \mathcal{L}) > 0$ for any converging sequence of points $p_n \to p$. That makes it possible to apply the lemma on almost minimum (6.4.8) to the function $p \mapsto \mathrm{ComRad}(p, \mathcal{L})$.

According to short hinge lemma (6.4.6), it is enough to show that

$$s_0 = \inf_{p\in\mathcal{L}} \mathrm{ComRad}(p, \mathcal{L}) \geqslant \varpi^\kappa \quad \text{for any} \quad p \in \mathcal{L}. \qquad ❼$$

We argue by contradiction, assuming that ❼ does not hold.

The rest of the proof is easier for geodesic spaces and yet easier for compact spaces. Thus we give three different arguments for each of these cases.

End of the proof for a compact space. By theorem 6.4.2-3 and compactness of $\mathcal{L}$, it's immediate that $s_0 > 0$. Take a point $p^* \in \mathcal{L}$ such that $r^* = \mathrm{ComRad}(p^*, \mathcal{L})$ is sufficiently close to s_0 (p^* such that $s_0 \leqslant r^* < \min\{\varpi^\kappa, \frac{3}{2}\cdot s_0\}$ will do). Then key lemma (6.4.7) applied for p^* and ℓ slightly bigger than r^* (say, such that $r^* < \ell < \min\{\varpi^\kappa, \frac{3}{2}\cdot s_0\}$) implies that

$$\measuredangle[x\,{}^{p^*}_q] \geqslant \tilde\measuredangle^\kappa(x\,{}^{p^*}_q)$$

for any hinge $[x\,{}^{p^*}_q]$ such that $|p^*x| + |xq| < \ell$. Thus $r^* \geqslant \ell$, a contradiction.

End of the proof for a geodesic space. Fix $\varepsilon = 0.0001$. Apply lemma on almost minimum (6.4.8) to find a point $p^* \in \mathcal{L}$ such that $r^* = \mathrm{ComRad}(p^*, \mathcal{L}) < \varpi^\kappa$ and

$$\mathrm{ComRad}(q, \mathcal{L}) > (1 - \varepsilon)\cdot r^* \qquad ❽$$

for any $q \in \overline{\mathrm{B}}(p^*, \frac{1}{\varepsilon}\cdot r^*)$.

Applying key lemma (6.4.7) for p^* and ℓ slightly bigger than r^* leads to a contradiction.

End of the proof, general case. Let us construct $p^* \in \mathcal{L}$ as in the previous case. Since $\mathcal{L}$ is not geodesic, we can not apply the key lemma directly. Instead, let us pass to the ultraproduct $\mathcal{L}^\omega$ which is a geodesic space (see 5.3.3).

According to theorem 6.4.2, inequality ❽ implies that the condition 6.4.2-1 holds for some fixed $R_1 = \frac{r^*}{100} > 0$ at any point $q \in \overline{\mathrm{B}}(p^*, \frac{1}{2\cdot\varepsilon}\cdot r^*) \subset \mathcal{L}$. Therefore a similar statement is true in the ultraproduct $\mathcal{L}^\omega$; i.e. for any point

$q_\omega \in \overline{\mathrm{B}}(p^*, \frac{1}{2\cdot\varepsilon}\cdot r^*) \subset \mathcal{L}^\omega$, condition 6.4.2-1 holds for say $R_1 = \frac{r^*}{101}$. In particular, $\mathrm{curv}_{q_\omega}\mathcal{L}^\omega \geqslant \kappa$ for any $q_\omega \in \overline{\mathrm{B}}(p^*, \frac{1}{2\cdot\varepsilon}\cdot r^*) \subset \mathcal{L}^\omega$.

Note that $r^* \geqslant \mathrm{ComRad}(p^*, \mathcal{L}^\omega)$. Therefore we can apply the lemma on almost minimum at point p^* for space $\mathcal{L}^\omega$ with function $x \mapsto \mathrm{ComRad}(x, \mathcal{L}^\omega)$ and $\varepsilon' = \sqrt{\varepsilon} = 0.01$.

For the obtained point $p^{**} \in \mathcal{L}^\omega$, we have $r^{**} = \mathrm{ComRad}(p^{**}, \mathcal{L}) < \varpi^\kappa$ and $\mathrm{ComRad}(q_\omega, \mathcal{L}^\omega) > (1-\varepsilon')\cdot r^{**}$ for any $q_\omega \in \overline{\mathrm{B}}(p^{**}, \frac{1}{\varepsilon'}\cdot r^{**})$. Thus applying key lemma (6.4.7) for p^{**} and ℓ slightly bigger than r^{**} leads to a contradiction. □

6.5 Properties of geodesics and angles

Remark. All proofs in this section can be easely modified to use only the local definition of CBB-spaces without use of the globalization theorem (6.4.4).

6.5.1. Geodesics do not split. *In* CBB*-space, geodesics do not bifurcate.*

More precisely: let $\mathcal{L} \in \mathrm{CBB}$ *and* $[px]$, $[py]$ *be two minimizing geodesics, then*

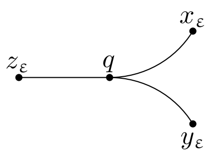
a) If there is an $\varepsilon > 0$*, such that* $\gamma_{[px]}(t) = \gamma_{[py]}(t)$ *for all* $t \in [0, \varepsilon)$*, then* $[px] \subset [py]$ *or* $[py] \subset [px]$*.*

b) If $\measuredangle[p\,{}^x_y] = 0$ *then* $[px] \subset [py]$ *or* $[py] \subset [px]$*.*

6.5.2. Corollary. *Let* $\mathcal{L} \in \mathrm{CBB}$*. Then restriction of any minimal geodesic in* $\mathcal{L}$ *to a proper segment is the unique minimal geodesic joining its endpoints.*

In case $\kappa \leqslant 0$, the proof is easier, since the model triangles are always defined. To deal with $\kappa > 0$ we have to do everything locally.

Proof of 6.5.1; a. Let $t_{\max}$ be the maximal value such that $\gamma_{[px]}(t) = \gamma_{[py]}(t)$ for all $t \in [0, t_{\max})$. Since geodesics are continuous $\gamma_{[px]}(t_{\max}) = \gamma_{[py]}(t_{\max})$, set



$$q = \gamma_{[px]}(t_{\max}) = \gamma_{[py]}(t_{\max}).$$

We have to show that $t_{\max} = \min\{|px|, |py|\}$.

If that is not true, choose sufficiently small $\varepsilon > 0$ such that points

$$x_\varepsilon = \gamma_{[px]}(t_{\max} + \varepsilon) \text{ and } y_\varepsilon = \gamma_{[py]}(t_{\max} + \varepsilon)$$

are distinct. Set

$$z_\varepsilon = \gamma_{[px]}(t_{\max} - \varepsilon) = \gamma_{[py]}(t_{\max} - \varepsilon).$$

Clearly, $\tilde\measuredangle^\kappa(q\,{}^{z_\varepsilon}_{x_\varepsilon}) = \tilde\measuredangle^\kappa(q\,{}^{z_\varepsilon}_{y_\varepsilon}) = \pi$. Thus from (1+3)-point comparison (6.1.2), $\tilde\measuredangle^\kappa(q\,{}^{x_\varepsilon}_{y_\varepsilon}) = 0$ and thus $x_\varepsilon = y_\varepsilon$, a contradiction.

(b). From hinge comparison 6.2.1d

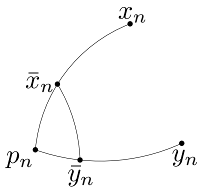
$$\measuredangle[p\,{}^x_y] = 0 \quad \Rightarrow \quad \tilde\measuredangle^\kappa\left(p\,{}^{\gamma_{[px]}(t)}_{\gamma_{[py]}(t)}\right) = 0$$

and thus $\gamma_{[px]}(t) = \gamma_{[py]}(t)$ for all small t. Therefore we can apply (a). □

6.5.3. Angle semicontinuity. *Let* $\mathcal{L}_n \in \mathrm{CBB}[\kappa]$ *and* $\mathcal{L}_n \xrightarrow{\omega} \mathcal{L}_\omega$*. Assume that sequence of hinges* $[p_n\,{}^{x_n}_{y_n}]$ *in* $\mathcal{L}_n$ *converges to a hinge* $[p_\omega\,{}^{x_\omega}_{y_\omega}]$ *in* $\mathcal{L}_\omega$*. Then*

$$\measuredangle[p_\omega\,{}^{x_\omega}_{y_\omega}] \leqslant \omega\text{-}lim\,\measuredangle[p_n\,{}^{x_n}_{y_n}].$$

Proof. From 6.2.2,



$$\measuredangle[p_\omega\,{}^{x_\omega}_{y_\omega}] = \sup_{\bar x_\omega, \bar y_\omega} \left\{ \tilde\measuredangle^\kappa(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}) \;\middle|\; \bar x_\omega \in\,]p_\omega x_\omega]\,,\ \bar y_\omega \in\,]p_\omega x_\omega] \right\}.$$

For fixed $\bar x_\omega \in\,]p_\omega x_\omega]$ and $\bar y_\omega \in\,]p_\omega x_\omega]$, choose $\bar x_n \in\,]px_n]$ and $\bar y_n \in\,]py_n]$ so that $\bar x_n \xrightarrow{\omega} \bar x_\omega$ and $\bar y_n \xrightarrow{\omega} \bar y_\omega$. Clearly

$$\tilde\measuredangle^\kappa(p_n\,{}^{\bar x_n}_{\bar y_n}) \xrightarrow{\omega} \tilde\measuredangle^\kappa(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}).$$

From the hinge comparison (6.2.1d), $\measuredangle[p_n\,{}^{x_n}_{y_n}] \geqslant \tilde\measuredangle^\kappa(p_n\,{}^{\bar x_n}_{\bar y_n})$. Hence the result. □

6.5.4. First variation formula. *Let $\mathcal{L} \in$ CBB. For any point q and geodesic $[px]$ in $\mathcal{L}$, we have*

$$|q\gamma_{[px]}(t)| = |qp| - t\cdot\cos\varphi + o(t), \qquad ❶$$

where φ is the infimum of angles between $[px]$ and all geodesics from p to q in the ultraproduct $\mathcal{L}^\omega$.

Remark. If $\mathcal{L}$ is proper space then $\mathcal{L}^\omega = \mathcal{L}$ and φ is achieved on some particular geodesic from p to q (see section 5.3).

As a corollary we obtain the following classical result:

6.5.5. Lemma about strong angle. *Let $\mathcal{L} \in$ CBB and $p, q \in \mathcal{L}$ be such that there is unique geodesic from p to q in the ultraproduct $\mathcal{L}^\omega$. Then for any hinge $[p\,{}^q_x]$ we have*

$$\measuredangle[p\,{}^q_x] = \lim_{\substack{\bar x \to p \\ \bar x \in [px]}} \tilde\measuredangle^\kappa(p\,{}^q_{\bar x}). \qquad ❷$$

for any $\kappa \in \mathbb{R}$ such that $|pq| < \varpi^\kappa$.

In partiqular ❷ holds if $p \in \operatorname{Str}(q)$ as well as if $q \in \operatorname{Str}(p)$

Remark.

- The above lemma is essentially due to Alexandrov. The right hand side in ❷ is called *strong angle* of hinge $[p\,{}^q_x]$. Note that in general metric space angle might differ from strong angle for a hinge.
- As it follows from ???, if there is a unique geodesic $[pq]$ in the ultraproduct $\mathcal{L}^\omega$ then $[pq]$ lies in $\mathcal{L}$.

Proof lemma about strong angle. The first statement follows directly from the first variation formula (6.5.4) and definition of model angle (see section 3.2). The second statement follows from the Plaut's theorem (6.1.6) applyied to $\mathcal{L}^\omega$. (Note that according to proposition 6.1.3 $\mathcal{L}^\omega \in$ CBB.) □

Proof of the first variation formula (6.5.4). Let $\mathcal{L} \in \mathrm{CBB}[\kappa]$. Without loss of generality we can assume that $\kappa \leqslant 0$. The inequality

$$|q\gamma_{[px]}(t)| \leqslant |qp| - t\cdot\cos\varphi + o(t)$$

is an immediate consequence of the first variation inequality (3.3.3). Thus, it is enough to show that

$$|q\gamma_{[px]}(t)| \geqslant |qp| - t\cdot\cos\varphi + o(t).$$

Assume the contrary, then there is $\varepsilon > 0$, such that $\varphi+\varepsilon < \pi$ and for a sequence $t_n \to 0+$ we have

$$|q\gamma_{[px]}(t_n)| < |qp| - t_n\cdot\cos(\varphi - \varepsilon). \quad ❸$$

Set $x_n = \gamma_{[px]}(t_n)$. Clearly $\tilde\measuredangle^\kappa(p\,{}^q_{x_n}) > \varphi + \frac{\varepsilon}{2}$ for all large n. Applying both parts of hinge comparison (6.2.1d), we get $\measuredangle[p\,{}^q_x] < \varphi - \frac{\varepsilon}{2}$ for all large n.

Assume $\mathcal{L}$ is geodesic. Choose a sequence of minimizing geodesics $[x_nq]$. Let $[x_nq] \xrightarrow{\omega} [pq]_{\mathcal{L}^\omega}$ (in general $[pq]$ might lie in $\mathcal{L}^\omega$). According to 6.5.3, the angle between $[pq]$ and $[px]$ is at most $\varphi - \frac{\varepsilon}{2}$, a contradiction.

Finally, if $\mathcal{L}$ is not geodesic choose a sequence $q_n \in \mathrm{Str}(x_n)$, such that $q_n \to q$ and the inequality $\tilde\measuredangle^\kappa(x_n\,{}^p_{q_n}) > \varphi + \frac{\varepsilon}{2}$ still holds. Then the same argument as above shows that $[x_nq_n]$ ω-converges to a geodesic $[pq]_{\mathcal{L}^\omega}$ from p to q having angle at most $\varphi - \frac{\varepsilon}{2}$ with $[px]$. □

6.6 On positive lower bound

The following lemma is a simple but useful property of positively curved spaces.

6.6.1. Lemma. *Let $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$ with $\kappa > 0$. Let $p \in \mathcal{L}$ then $|p*|\colon \mathcal{L} \to \mathbb{R}$ is concave in* $\mathrm{B}(p, \varpi^\kappa)\backslash\mathrm{B}(p, \frac{\varpi^\kappa}{2})$.

Proof. This is an immediate consequence of 6.3.1b. □

The following theorem states that if one ignores a few exceptional spaces, then the diameter of a space with positive lower curvature bound is bounded. Note that many authors (but not us) exclude these spaces in the definition of Alexandrov space with positive lower curvature bound.

6.6.2. On diameter of space. *Let $\kappa > 0$ and $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$. Then either*

a) $\operatorname{diam}\mathcal{L} \leqslant \varpi^\kappa$;

b) or $\mathcal{L}$ is isometric to one of the following exceptional spaces:

1. *real line* $\mathbb{R}$,
2. *a ray* $\mathbb{R}_{\geqslant}$,
3. *a closed interval* $[0, a] \in \mathbb{R}$, $a > \varpi^\kappa$,
4. *a circle* $\mathbb{S}^1_a$ *of length* $a > 2\cdot\varpi^\kappa$.

Proof. Using rescaling, we can assume that $\kappa = 1$ and thus $\varpi^\kappa = \pi$.

Assume that $\mathcal{L}$ is a geodesic space and $\operatorname{diam}\mathcal{L} > \pi$. Choose $x, y \in \mathcal{L}$ so that $|xy| = \pi+\varepsilon$, $0 < \varepsilon < \frac{\pi}{4}$. By moving y slightly, we can also assume that geodesic $[xy]$ is unique; to prove it, use either Plaut's theorem (6.1.6) or that geodesics do not split (6.5.1). Let z be the mid-point of a geodesic $[xy]$.

Consider the function $f = |x*| + |y*|$. As it follows from lemma 6.6.1, f is concave in $\mathrm{B}(z, \frac{\varepsilon}{4})$. Let $p \in \mathrm{B}(z, \frac{\varepsilon}{4})$. Choose a shortest geodesic $[zp]$. Then $g(t) = f \circ \gamma_{[zp]}(t)$ is concave. From corollary 6.2.3, we have that $g^+(0) = 0$. Therefore g is nonincreasing which means that

$$|xp| + |yp| = g(|zp|) \leqslant g(0) = |xz| + |yz| = |xy|.$$

Since the geodesic $[xy]$ is unique this means that $p \in [xy]$ and hence $\mathrm{B}(z, \frac{\varepsilon}{4})$ only contains points of $[xy]$. Since in CBB-spaces, geodesics do not bifurcate

(6.5.1a), it follows that all of $\mathcal{L}$ coincides with the maximal extension of $[xy]$ as a local geodesic γ (which might be not a minimizing geodesic). In other words, $\mathcal{L}$ is isometric to a 1-dimensional Riemannian manifold with possibly non empty boundary. From this, it is easy to see that $\mathcal{L}$ falls into one of the exceptional spaces described in the theorem.

Lastly, if $\mathcal{L}$ is not geodesic and $\operatorname{diam}\mathcal{L} > \pi$ then the above argument applied to $\mathcal{L}^{\circledcirc}$ yields that $\mathcal{L}^{\circledcirc}$ is isometric to one of the exceptional spaces. As all of those spaces are proper it means that $\mathcal{L} = \mathcal{L}^{\circledcirc}$. □

6.6.3. On perimeter of triangle. *Let $\kappa > 0$, $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$ and $\operatorname{diam}\mathcal{L} \leqslant \varpi^\kappa$. Then perimeter of any triple of points $p, q, r \in \mathcal{L}$ is at most $2{\cdot}\varpi^\kappa$.*

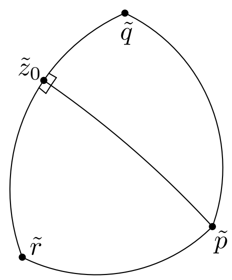
Proof. We argue by contradiction. Suppose $|pq| + |qr| + |rp| > 2{\cdot}\varpi^\kappa$ for some $p, q, r \in \mathcal{L}$. Since $\mathcal{L} \in \mathrm{CBB}\lfloor K\rfloor$ for any $0 < K < \kappa$ (6.4.5), by making κ slightly smaller we can assume that $\operatorname{diam}\mathcal{L} < \varpi^\kappa$.

Further, by rescaling can assume that $\kappa = 1$, so $\varpi^\kappa = \pi$ and $\mathbb{M}^2[\kappa] = \mathbb{S}^2$.

Since $\mathcal{L}$ is G_δ-geodesic (6.1.6), it is sufficient to consider case when there is a geodesic $[qr]$.

First note that since $\operatorname{diam}\mathcal{L} < \pi$, by 6.3.1$b$ we have that $y(t) = \operatorname{md}^1|p\gamma_{[qr]}(t)|$ satisfies the differential inequality $y'' \leqslant 1 - y$.

Take $z_0 \in [qr]$ so that restriction $|p{*}||_{[qr]}$ attains its maximum at z_0, set $t_0 = |qz_0|$ so $z_0 = \gamma_{[qr]}(t_0)$. Consider the following model configuration: two minimizing geodesics $[\tilde p\tilde z_0]$, $[\tilde q\tilde r]$ in $\mathbb{S}^2$ such that



$$|\tilde p\tilde z_0| = |pz_0|,\ \ |\tilde q\tilde r| = |qr|,\ \ |\tilde z_0\tilde q| = |z_0q|,\ \ |\tilde z_0\tilde r| = |z_0q|$$

and

$$\measuredangle[\tilde z_0\,{}^{\tilde q}_{\tilde p}] = \measuredangle[\tilde z_0\,{}^{\tilde r}_{\tilde p}] = \tfrac{\pi}{2}.$$

Clearly, $\bar y(t) = \operatorname{md}^1|\tilde p\gamma_{[\tilde q\tilde r]}(t)|$ satisfies $\bar y'' = 1 - \bar y$ and $\bar y'(t_0) = 0$, $\bar y(t_0) = y(t_0)$. Since z_0 is a maximum point, $y(t) \leqslant y(t_0) + o(t - t_0)$; thus, $\bar y(t)$ is a barrier for $y(t) = \operatorname{md}^1|p\gamma_{[qr]}(t)|$ at t_0 by 4.2.1c. From barrier inequality (4.2.1c), we get

$$|\tilde p\gamma_{[\tilde q\tilde r]}(t)| \geqslant |p\gamma_{[qr]}(t)|,$$

thus, $|\tilde p\tilde q| \geqslant |pq|$ and $|\tilde p\tilde r| \geqslant |pr|$. Thus, $|pq| + |qr| + |rp|$ can not exceed the perimeter of the spherical triangle $[\tilde p\tilde q\tilde r]$, therefore

$$|pq| + |qr| + |rp| \leqslant 2{\cdot}\pi.$$

□

Let $\kappa > 0$. Consider the following extension $\tilde\measuredangle^{\kappa+}(*\,{}^*_*)$ of the model angle function $\tilde\measuredangle^{\kappa}(*\,{}^*_*)$. Some authors define camparison angle to be $\tilde\measuredangle^{\kappa+}(*\,{}^*_*)$ but that leads to two different definition of model angles — one for CBB-spaces and an other for CBA-spaces.

6.6.4. Definition of extended angle. *Let p, q, r be points in a metric space such that $p \ne q$, $p \ne r$ then set*

$$\tilde\measuredangle^{\kappa+}(p\,{}^q_r) = \sup\left\{\, \tilde\measuredangle^{K}(p\,{}^q_r) \;\middle|\; K \leqslant \kappa \,\right\}.$$

The following characterization of extended angles is an easy consequence of the definition

6.6.5. Proposition. *Let p, q, r be points in a metric space such that $p \neq q$, $p \neq r$ then*

a) $\tilde\measuredangle^{\kappa+}(p\,{}^q_r) = \tilde\measuredangle^{\kappa}(p\,{}^q_r)$ *once* $\tilde\measuredangle^{\kappa}(p\,{}^q_r)$ *is defined;*
b) $\tilde\measuredangle^{\kappa+}(p\,{}^q_r) = \tilde\measuredangle^{\kappa+}(p\,{}^r_q) = 0$ *if* $|pq| + |qr| = |pr|$*;*
c) $\tilde\measuredangle^{\kappa+}(p\,{}^q_r) = \pi$ *if none of above is applicable.*

6.6.6. Extended angle comparison. *Let $\kappa > 0$ and $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$. Then for any hinge $[p\,{}^q_r]$ we have $\measuredangle[p\,{}^q_r] \geqslant \tilde\measuredangle^{\kappa+}(p\,{}^q_r)$.*

Proof. From 6.4.5, $K < \kappa$ implies $\mathrm{CBB}\lfloor K\rfloor \supset \mathrm{CBB}\lfloor\kappa\rfloor$; thus the extended angle comparison follows from the definition. □

6.7 Comments and open problems

The question whether the first part of 6.2.1d suffices to conclude that $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$ is a long standing open problem (possibly dating back to Alexandrov), but as far as we know, it was first formally stated in print in [BBI 01] (see the footnote in subsection 4.1.5).

6.7.1. Open question. *Let $\mathcal{L}$ be a complete geodesic space (you can also assume that $\mathcal{L}$ is homeomorphic to $\mathbb{S}^2$) such that for any hinge $[x\,{}^p_y]$ in $\mathcal{L}$, the angle $\measuredangle[x\,{}^p_y]$ is defined and*

$$\measuredangle[x\,{}^p_y] \geqslant \tilde\measuredangle^0(x\,{}^p_y).$$

Is it true that $\mathcal{L} \in \mathrm{CBB}\lfloor 0\rfloor$?

6.8 Exercises

1. (Wald's definition) Let $\mathcal{L}$ be a complete intrinsic space and $\kappa \leqslant 0$. Prove that $\mathcal{L} \in \mathrm{CBB}\lfloor\kappa\rfloor$ if and only if for any quadruple of points $p, q, r, s \in \mathcal{L}$ can be isometrically embedded into some $\mathbb{M}^2[K]$ for $K \geqslant \kappa$.
   Is the same true for $\kappa > 0$, what is the difference?
2. (a) Show that any space $\mathcal{L} \in \mathrm{CBB}\lfloor 0\rfloor$ satisfies the following condition:
   - ⋄ For any three points $p, q, r \in \mathcal{L}$, if q' and r' are mid-points of geodesics $[pq]$ and $[pr]$ correspondingly then $2{\cdot}|q'r'| \geqslant |qr|$.

   (b) Construct a geodesic space, which satisfies above condition, but which is not an Alexandrov space.
3. Assume $\mathcal{X}$ is a complete intrinsic space which satisfy the following condition, any 4-point subset can be isometrically embedded in Euclidean 3-space.
   Prove that $\mathcal{X}$ is isometric to a closed convex substet of a Hilbert space.

4. (a) Construct a geodesic space $\mathcal{X}$ which is not an Alexandrov space, but which meets the following condition: for any 3 points $p, x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $z \in]xy[$

$$\tilde{\measuredangle}^0(z\,{}^p_x) + \tilde{\measuredangle}^0(z\,{}^p_y) \leqslant \pi.$$

(Compare with the construction of Urysohn's space [Gromov, $3.11\frac{3}{2}{}_+$].)
(b) Let us change condition a bit: for any 2 points $x, y \in \mathcal{X}$ there is a geodesic $[xy]$ such that for any $p \in \mathcal{X}$ and $z \in]xy[$

$$\tilde{\measuredangle}^0(z\,{}^p_x) + \tilde{\measuredangle}^0(z\,{}^p_y) \leqslant \pi.$$

Show that $\mathcal{X} \in \mathrm{CBB}\lfloor 0 \rfloor$
5. ???Assume $\mathcal{L} \in \mathrm{CBB}\lfloor \kappa \rfloor$, $K > \kappa$ and $\mathrm{curv}_p\,\mathcal{L} \geqslant K$ for all p in a dense G_δ-set of $\mathcal{L}$. Prove that $\mathcal{L} \in \mathrm{CBB}\lfloor K \rfloor$
6. Construct a space $\mathcal{L} \in \mathrm{CBB}\lfloor 0 \rfloor$ which is not geodesic.
7. Let $\mathcal{L}$ be a Minkowski space (i.e. a metric space with underlying set $\mathbb{R}^m$ and metric induced by a norm). Show that $\mathcal{L} \in \mathrm{CBB}$ if and only if $\mathcal{L} \stackrel{iso}{=\!=} \mathbb{E}^m$.