(1) Let $M^n \subset \mathbb{R}^N$ be a smooth manifold with the induced metric Riemannian metric g. Let $\bar{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^N$. Let X, Y be smooth tangent vector fields on M and let $\bar{X}, \bar{Y}$ be their smooth extensions to an open neighbourhood U of M. We defined $\nabla_X Y$ by the formula

$$\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^t$$

It was shown in class that this defines a well-defined connections on M which is Riemannian with respect to g.

Prove that ∇ defined this way is torsion free and conclude that it coincides with the Levi-Civita connection with respect to g.

(2) Give a careful the proof of the following result from class Let $\phi\colon (M^n, g_1) \to (N^n, g_2)$ be an isometry. Given a connection ∇ on M define the push-forward connection $\phi_\star(\nabla)$ on B by the formula $\phi_\star(\nabla)_{\phi_\star X} \phi_\star Y = \phi_*(\nabla_X Y)$

(a) Prove that push forward of the Levi-Civita connection on M is the Levi-Civita connection on N.

(b) Prove that if $\gamma(t)$ is a geodesic in M then $\phi(\gamma(t))$ is a geodesic in N.