(1) Consider the Riemannian metric on $SO(n)$ induced from the standard inclusion $SO(n) \subset \mathbb{R}^{n^2}$.

Prove that this metric is biinvarinat.

Hint: Show that the standard inner product on the set of all $n \times n$ matrices identified with $\mathbb{R}^{n^2}$ is given by $\langle A, B \rangle = tr(A \cdot B^t)$.

(2) Let G be a Lie group. Let X be a left invariant vector field and let $\gamma(t)$ be its integral curve with $\gamma(0) = e$.
   (a) Prove that $\lambda(t) = g \cdot \gamma(t)$ is also an integral curve of X for any $g \in G$. Conclude that the flow ϕ_t of X is given by $\phi_t(g) = g \cdot \gamma(t)$.
   (b) Show that $\gamma(t+s) = \gamma(t) \cdot \gamma(s)$ where defined.
   (c) Show that ϕ_t is defined for all $t \in \mathbb{R}$.

(3) Prove that a Riemannian metric g on a manifold M is smooth if and only if $\langle X, Y \rangle(p)$ is smooth for any smooth vector fields X, Y.

(4) Consider the immersion $\phi\colon S^2 \to \mathbb{R}^4$ from the previous homework. Recall that F is given by the formula $F(x, y, z)) = (x^2 - y^2, xy, xz, yz)$. Consider the Riemannian metric g on S^2 induced by F. That is, for any $p \in S^2, u, v \in T_P S^2$ set $\langle u, v \rangle_g \overset{def}{:=} \langle dF_p(u), dF_p(v) \rangle$.

Find $g_{ij}(0,0)$ for the standard upper hemisphere parameterization $\phi(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$