(1) Let $f\colon M \to N$ be a smooth map between smooth manifolds. Let $p \in M$ and let $v \in T_pM$ be a tangent vector at p.

Recall that we had two different definitions of $df_p(v)$.

The first definition was as follows: take a smooth curve γ in M representing v (i.e. $\gamma'(0) = v$) and define $df_p(v)$ to be $(f \circ \gamma)'(0)$.

The other definition was by viewing v as a derivation $\mathcal{D}$ at p. We then define $df_p(\mathcal{D})$ to be a derivation at $f(p)$ given by the formula $df_p(\mathcal{D})(g) = \mathcal{D}(g \circ f)$ for any smooth $g\colon N \to \mathbb{R}$.

Prove that these two definitions are equivalent.

Hint: Choose local coordinates on M and N and check that both maps are given by the same matrix with respect to these coordinates.

(2) Let $f\colon M \to N$ be a diffeomorphism between smooth manifolds. Let X, Y be vector fields on M.

Prove that $[f_*(X), f_*(Y)] = f_*([X, Y])$.

(3) Prove that $O(n)$ is a smooth manifold and compute its dimension.

Hint: Use that orthogonal matrices satisfy $A \cdot A^t = Id$ to write $O(n)$ as a level set $\{f = c\}$ for some smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with c being a regular value of f.